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On $(\alpha')^2$ corrections to the D-brane action for non-geodesic world-volume embeddings

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Abstract: In `hep-th/9903210` (curvature)^2 terms of the effective D-brane action were derived to lowest order in the string coupling. Their results are correct up to ambiguous terms which involve the second fundamental form of the D-brane. We compute five point string amplitudes on the disk. We compare the subleading order in α' of the string amplitudes with the proposed lagrangian of `hep-th/9903210` supplemented by the ambiguous terms. The comparison determines the complete form of the gravitational terms in the effective D-brane action to order $\mathcal{O}(\alpha'^{\,2})$. Our results are valid for arbitrary ambient geometries and world-volume embeddings.

Keywords: Superstrings and Heterotic Strings, D-branes.

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1. Introduction

The D-brane low energy effective action for a single brane describes the field theory of a gauge boson A_α and scalar fields Y^i, as well as their interactions with the closed string modes of the bulk [1]–[4] (for reviews on D-branes see [6, 7]). For a system of N coincident D-branes the low energy effective action is generalized to a non-abelian theory with gauge group U(N). Since we have a supersymmetric theory there is of course a fermionic analog of these actions but we will not deal with it in this paper.

A low energy effective action is obtained in field theory by integrating out massive modes above some energy scale. This modifies the full action of the theory by introducing higher derivative terms which encode the effect of integrating out the massive modes. Effective actions have a range of validity up to the scale above which we integrate out. The D-brane low energy effective action is the action of the massless modes. Obviously it receives higher order corrections. Therefore, for energies much smaller than the scale of the massive string modes $1/\alpha'$, the effective action is given as an expansion in powers of α' the fundamental energy scale in string theory.

In the case of the D-brane the low energy effective action has two parts. The parity even part is given by the Dirac-Born-Infeld (DBI) [8, 9] action and the parity odd by the Wess-Zumino (WZ) [3, 11, 12, 13] action. The Wess-Zumino action describes the coupling of D-branes to Ramond-Ramond fields. The form of this action is determined by requiring cancellation of the chiral gauge anomaly from the field theory on the brane against the gravitational anomaly from the bulk for intersecting D-branes [11, 12, 13]. The WZ term takes the form:

$$\int_{M^{(p+1)}} \mathcal{L}_{WZ}^{(p)} = T_{(p)} \int_{M^{(p+1)}} C \wedge \mathrm{tr}_N \left(e^{2\pi\alpha' F} \right) \wedge \left(\frac{\hat{A}(4\pi^2\alpha' R_T)}{\hat{A}(4\pi^2\alpha' R_N)} \right)^{1/2} , \tag{1.1}$$

where $C = C^{(0)} + C^{(1)} + \cdots C^{(9)}$ are the R-R p-form potentials pulled-back on the D-brane with odd forms contributing in the IIA theory and even forms in the IIB theory. The trace of the field strength $F_{\alpha\beta}$ is over the U(N) gauge indices for N coincident D-branes. $\hat{A}$ is the Dirac "roof" genus and its square root has the expansion:

$$\sqrt{\hat{A}(R)} = 1 - \frac{1}{48} p_1(R) + \frac{1}{2560} p_1^2(R) - \cdots . \tag{1.2}$$

Another method of finding these couplings is by considering scattering of gravitons on the D-branes in the boundary state formalism [14]–[17].

The DBI action:

$$S_{DBI} = T_{(p)} \int dx^{p+1} e^{-\phi} \sqrt{\det \left((G_{\mu\nu} + B_{\mu\nu}) \partial_\alpha X^\mu \partial_\beta X^\nu + 2\pi\alpha' F_{\alpha\beta} \right)} \tag{1.3}$$

describes the coupling of the brane modes to the NS-NS sector bulk fields: $\phi, g_{\mu\nu}, B_{\mu\nu}$. The Born-Infeld action for a single brane in flat background has an expansion which for $\alpha' \to 0$ reduces to the U(1) Yang-Mills theory. For N coincident D-branes the DBI becomes a non-abelian field theory whose $\alpha' \to 0$ limit is an SU(N) SYM theory. The expansion includes also terms of the form $(F)^n$ which give contact interactions of n particle scattering in the effective field theory. The DBI action should be supplemented by higher derivative corrections, in the sense that they vanish for F constant. In the curved brane case, which is the main interest of this paper, derivative corrections involve pull-backs of derivatives of the NS-NS background fields and $\partial^n(\partial_\alpha X^\mu)$ corrections, where $\partial_\alpha X^\mu$ the embedding of the brane in the ambient spacetime; all of which vanish for constant background and embedding. The issue of finding the form of these derivative corrections has been addressed in [18, 19]. In [18] α'^2 corrections to the DBI for non-constant backgrounds were determined, while in [19] the α'^2 corrections involve derivatives of the gauge field strength F and of the embedding ∂X.

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There are several ways to compute corrections to the D-brane effective action. One of them involves computing renormalization group beta functions for the field theory of strings on the world-sheet. Consistency conditions (superconformal invariance) impose that these beta functions should vanish. From these conditions we can find equations of motion for the background fields. This method was used to derive the Born-Infeld action in [8].

Another way to determine effective actions is by expanding string amplitudes in powers of the string scale α' and looking for terms in the effective action to reproduce this expansion. We use this method for higher order derivative corrections since the beta function computation is quite more involved requiring amplitudes with at least three loops. This method has also been used to check the validity of the Born-Infeld action. It has been found through such amplitude computations that the non-abelian Born-Infeld encounters problems when expanded beyond the leading order in α' [10]. However, in this paper we will focus on the higher α' corrections to the single (abelian) D-brane action. In [18] α'^2 corrections to DBI were found by comparison to the α' expansion of the four-point functions computed in [20]. This way they arrived to the following langrangian up to order α'^2:

$$L^{(p)} = T_p\, e^{-\phi}\sqrt{\tilde{g}}\Big[1 - \frac{1}{24}\frac{(4\pi^2\alpha')^2}{32\pi^2}\Big((R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta} - 2(R_T)_{\alpha\beta}(R_T)^{\alpha\beta} - \\ - (R_N)_{\alpha\beta ij}(R_N)^{\alpha\beta ij} + 2\bar{R}_{ij}\bar{R}^{ij}\Big)\Big],$$

where R_T and R_N are the Riemann tensors constructed from the world-volume and normal bundle connections (see appendix B).

The corrections computed in [18] were ambiguous because certain combinations of the proposed derivative terms turned out to vanish for linear expansions in the fields. We will find out that there are five ambiguous terms in the D-brane action which cannot be fixed using the constraints deduced from the four-point functions computed in [18]. Three of these terms are total derivatives for linear expansion of the curvature and second fundamental form $\Omega^i_{\alpha\beta}$ tensors[1] and we denote their coefficients by c_1, c_2, c_3. The other two ambiguous terms give vanishing contribution to the four-point functions because they involve tensors which vanish due to the lowest order equations of motion and we denote their coefficients by c_4, c_5. The ambiguous terms can be summarized by the langrangian:

$$L^{\rm ambig} = \Big[c_1\Big(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta} + \hat{R}^2\Big) + \\ + c_2\Big(4R_{\alpha\beta\gamma\delta}(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta}) - 8\hat{R}_{\alpha\beta}(\Omega^{\alpha\gamma}\cdot\Omega_\gamma^{\ \beta}) + 2\hat{R}(\Omega^{\alpha\beta}\cdot\Omega_{\alpha\beta})\Big) + \\ + c_3\Big(2(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta}) - 2(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})(\Omega^{\alpha\delta}\cdot\Omega^{\beta\gamma}) -$$

[1] See appendix B for definition of second fundamental form tensor.

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$$
\begin{aligned}
&- 4(\Omega_{\alpha\gamma}\cdot\Omega_{\beta}^{\gamma})(\Omega^{\alpha\gamma}\cdot\Omega_{\gamma}^{\beta}) + (\Omega^{\alpha\beta}\cdot\Omega_{\alpha\beta})(\Omega^{\gamma\delta}\cdot\Omega_{\gamma\delta})\Big) + \\
&+ c_4((\Omega_{\gamma}^{\gamma}\cdot\Omega_{\alpha}^{\beta})(\Omega_{\delta}^{\alpha}\cdot\Omega_{\beta}^{\delta})) + c_5\Big(\hat{R}_{\alpha\beta}(\Omega^{\alpha\beta}\cdot\Omega_{\gamma}^{\gamma})\Big)\Big].
\end{aligned}
$$

In order to determine these ambiguous terms one has to compute higher point amplitudes. In this paper we will extend the analysis of [18] to determine the complete form of the α'^2 corrections to the DBI for curved backgrounds and non-trivial embeddings. We shall find out that the constraints derived from our five-point functions computations are sufficient to determine the ambiguous coefficients c_1, c_2, c_3 if we use in addition a duality argument described in [18]. The remaining coefficients c_4, c_5 will remain undetermined since they give vanishing contribution to the amplitudes we will consider. Our computations are at disk-level which is the lowest order of the perturbative expansion of the D-brane action in the string coupling constant.

In section 2 we give a short overview of the basic machinery for computing string amplitudes on the disc. In section 3 we compute the three scalar with one graviton string amplitude. In section 4 we compare the $\alpha' \to 0$ limit of the string amplitude computed in section 3 with the corresponding field theory expression from the DBI action in order to fix the normalization. This way we have a check of our computations as well. In section 5 we present the ambiguous terms of the D-brane action to order α'^2 and compute the contribution of the proposed lagrangian to the three scalar with one graviton scattering. Consequently we expand the string amplitude to subleading order in α' and compare with the field theory amplitude to obtain a relation among the c_2 and c_3 coefficients. We shall also find out that c_4 cannot be determined from this amplitude. We repeat the above analysis for the scattering of two gravitons with one scalar in sections 6 and 7 and find a similar relation among c_1 and c_2 as well as that c_5 cannot be determined with our methods. Subsequently we use a non-perturbative argument described in [18] to fix the value of the ambiguities c_1, c_2, c_3. At the end we are left with the ambiguities c_4, c_5 which are proportional to the lowest order equations of motion for the scalar fields and in section 8 we comment on this result and find a possible explanation of our inability to fix them. We show that such terms remain arbitrary because they can be eliminated by employing field redefinitions under which the string S-matrix is invariant [29]. The appendices contain integral formulas needed for the string amplitudes and a review of the geometrical characteristics of submanifolds used in constructing candidate derivative terms for the D-brane action.

2. Preliminaries

In space-time D-branes are represented as static p-dimensional defects. As a result of these defects we must impose different boundary conditions, on the world-sheet

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boundary, to coordinates tangent and normal to the D-brane.

$$\begin{aligned} \partial_\perp X^\alpha \mid_{\partial\Sigma} &= 0 \\ X^i \mid_{\partial\Sigma} &= 0\,. \end{aligned} \tag{2.1}$$

The lower case Greek indices, ($\alpha = 0, 1 \ldots, p$), correspond to directions parallel to the brane and the lower Latin ones, ($i = p+1, \ldots, 9$) to normal coordinates. These constrains are respectively Neumann and Dirichlet boundary conditions.

When calculating tree string amplitudes we evaluate the partition function on a world-sheet with the topology of a disc. Before presenting details of our calculation, we will review the basic formalism of string vertex operators and their expectation values on the disc. We follow closely the review of [22] and references therein [21, 23].

It turns out that it is convenient to introduce the conformally equivalent description of the disc on the upper half complex plane for string amplitude computations. We chose this representation of the disc because it is easier to evaluate string correlators on the half complex plane. We denote the upper half plane as $\mathcal{H}^+$ and using radial coordinates z on the half-complex plane, the real axis becomes the world-sheet boundary.

The string operators for an NS-NS massless closed string have the following general form

$$V(z, \bar{z}) = \epsilon_{\mu\nu} : V_s^\mu(z) :: V_s^\nu(\bar{z}) :\,, \tag{2.2}$$

where $\mu = 0, 1, \ldots, 9$ and $s = 0, -1$ denotes the superghost charge or equivalently the picture in which the operator is in. The total superghost charge on the disk is required to be $Q_{sg} = -2$ as a consequence of the requirement for superdiffeomorphism invariance. The holomorphic(left moving) parts are given by

$$\begin{aligned} V_{-1}^\mu(p, z) &= e^{-\phi(z)} \psi^\mu(z) e^{\imath p\cdot X(z)} \\ V_0^\mu(p, z) &= (\partial X^\mu(z) + \imath p \cdot \psi(z) \psi^\mu(z)) e^{\imath p\cdot X(z)} \end{aligned} \tag{2.3}$$

and similar expressions for the anti-holomorphic part. The expectation values of string vertices are found using the following correlators:

$$\begin{aligned} \langle X^\mu(z) X^\nu(w) \rangle &= -\eta^{\mu\nu} \log(z-w) \\ \langle \psi^\mu(z) \psi^\nu(w) \rangle &= -\frac{\eta^{\mu\nu}}{z-w} \\ \langle \phi(z) \phi(w) \rangle &= -\log(z-w)\,. \end{aligned} \tag{2.4}$$

Because of the boundary conditions we have non-trivial correlators between right and left moving strings

$$\begin{aligned} \langle X^\mu(z) \bar{X}^\nu(\bar{w}) \rangle &= -D^{\mu\nu} \log(z-\bar{w}) \\ \langle \psi^\mu(z) \bar{\psi}^\nu(\bar{w}) \rangle &= -\frac{D^{\mu\nu}}{z-\bar{w}} \\ \langle \phi(z) \bar{\phi}(\bar{w}) \rangle &= -\log(z-\bar{w})\,, \end{aligned} \tag{2.5}$$

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where D^μ_ν is a diagonal matrix with $+1$ for directions tangent to the world-volume and -1 for transverse directions. We raise and lower indices using $\eta^{\mu\nu}$ [20]. At this point it is useful to define two more projection matrices V^μ_ν and N^μ_ν which project on the tangent and normal to the brane spaces respectively. These projection matrices satisfy the following identities

$$\begin{aligned} D_{\mu\nu} &= V_{\mu\nu} - N_{\mu\nu} \\ \eta_{\mu\nu} &= V_{\mu\nu} + N_{\mu\nu} \,. \end{aligned} \tag{2.6}$$

By extending the definition of the fields to the whole complex plane [22] we can write all our vertices in terms of left moving string operators by making the substitutions

$$\bar{X}^\mu(\bar{z}) \to D^\mu_\nu X^\nu(\bar{z}) \qquad \bar{\psi}^\mu(\bar{z}) \to D^\mu_\nu \psi^\nu(\bar{z}) \qquad \bar{\phi}(\bar{z}) \to \phi(\bar{z})\,, \tag{2.7}$$

where $z \in \mathcal{H}^+$. This way we use standard correlators (2.4) in string amplitude computations. After these replacements we write the vertex operators as follows:

$$V(z,\bar{z}) = (\epsilon D)_{\mu\nu} : V^\mu_s(p,z) :: V^\nu_s(Dp,\bar{z}) : \,. \tag{2.8}$$

Similar manipulations allow us to write the open string vertex operators for Neumann and Dirichlet conditions [22]

$$\begin{aligned} V^\mu_{-1}(2k,z) &= e^{-\phi(z)}\psi^\mu(z)e^{\imath 2k\cdot X(z)} \\ V^\mu_0(2k,z) &= (\partial X^\mu(z) + \imath 2k\cdot\psi(z)\psi^\mu(z))e^{\imath 2k\cdot X(z)} \end{aligned} \tag{2.9}$$

with z on the real axis and the momentum k^μ is restricted to be tangent to the brane. The vertex operators (2.9) are with momenta $2k$ since we have set $\alpha' = 2$ even though they are open strings. The general expression of a string amplitude is

$$A = \int \frac{d^2z_J d^2x_I}{V_{CKG}} \left\langle \prod_{I=1}^{n} : V(x_I) : \prod_{J=1}^{m} : V(z_J,\bar{z}_J) : \right\rangle \,, \tag{2.10}$$

where n, m are the number of open and closed string operators, respectively. The diffeomorphisms group CKG (Conformal Killing Group) can be used to fix the position of n_K operators, where n_K the number of conformal killing vectors of the world-sheet surface. In fixing these positions we introduce n_K fermionic ghosts.

Therefore for the disc with $\mathrm{SL}(2,\mathbb{R})$ CKG we have to insert three c-ghosts in the fixed position operators. Their correlator is given by

$$\langle c(z_1)c(z_2)c(z_3)\rangle = C^{\text{ghost}}_{D_2}(z_1-z_2)(z_1-z_3)(z_2-z_3)\,, \tag{2.11}$$

where $C^{\text{ghost}}_{D_2}$ is a normalization constant [5]

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3. Graviton-three scalar scattering amplitude

In this section we will calculate the scattering amplitude of one graviton with 3 world-volume scalars. As explained in the introduction this amplitude expanded to subleading order in the momenta will help us determine some of the ambiguity of the α'^2 corrections computed in [18]. We explain more on these ambiguities in section 5. The graviton has momentum in the bulk space-time and the scalar fields can only propagate parallel to the p-brane. We insert one closed string vertex operator in the -1 picture and the open string vertices in the 0 picture. The amplitude has the following expression

$$A \sim \int d^2z \left\langle \prod_{I=1}^{3} : c(x_I) V_0^i(x_I) : : V_{-1}^{\mu}(z) : : V_{-1}^{\nu}(\bar{z}) : \right\rangle \zeta_{Ii} (\epsilon D)_{\mu\nu} + (2 \leftrightarrow 3) \,, \quad (3.1)$$

where the index $I = 1, 2, 3$ denotes the scalars. The exchange of the particles 2 and 3 in the last line is necessary since the $\mathrm{SL}(2, \mathbb{R})$ diffeomorphisms group does not change the cyclic ordering of the open string vertex operators on the disc boundary (real axis). For the graviton amplitude in the abelian case it turns out that this exchange results in a factor of two in our amplitude. For non-abelian cases and the Kalb-Ramond field this will have a non-trivial effect on the amplitude.

We use the following kinematic invariants to describe this scattering process:

$$\begin{aligned} s &= -4k_1k_2 \qquad t = -4k_1k_3 \qquad u = -4k_2k_3 \\ q^2 &= (Vp)^2 = -\left(\frac{s+t+u}{2}\right) , \end{aligned} \quad (3.2)$$

where k_1, k_2, k_3 are the three open string momenta and p^μ is the momentum of the closed string. To derive the last equation we used the conservation of momentum on the world-volume

$$(k_1 + k_2 + k_3 + (Vp))^\mu = 0 \,. \quad (3.3)$$

The usual gauge transversality conditions for the graviton and scalar fields translate to the following conditions on the momenta and polarizations:

$$\zeta_I^\mu k_{I\mu} = 0 \qquad \epsilon^{\mu\nu} p_\nu = 0 \qquad \epsilon_\mu^\mu = 0 \,. \quad (3.4)$$

We use the expressions from section 2 for the vertex operators to expand the expression appearing inside the integral in (3.1). The integrand in (3.1) becomes

$$\begin{aligned} &\Big\langle : c(x_1)(\partial X^i + 2i(k_1\cdot\psi)\psi^i)(x_1) e^{2ik_1\cdot X(x_1)} : : c(x_2)(\partial X^i + 2i(k_2\cdot\psi)\psi^i)(x_2) \times \\ &\quad \times e^{2ik_2\cdot X(x_2)} : : c(x_3)(\partial X^i + 2i(k_3\cdot\psi)\psi^i)(x_3) e^{2ik_3\cdot X(x_3)} : \times \\ &\quad \times : e^{-\phi(z)}\psi^\mu(z) e^{ip\cdot X(z)} : : e^{-\phi(\bar{z})}\psi^\nu(\bar{z}) e^{i(Dp)\cdot X(\bar{z})} : \Big\rangle \zeta_{1i}\zeta_{2j}\zeta_{3l}(\epsilon D)_{\mu\nu} + \\ &+ (2 \leftrightarrow 3) \,. \end{aligned} \quad (3.5)$$

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Expanding the terms in the amplitude we get four different types of path integrals we need to evaluate. We use the expressions from section 2 for the correlators to evaluate the expression appearing inside the integral in (3.5). Combining all terms we get

$$
\begin{aligned}
&\Bigg\{\eta^{\mu\nu}\Bigg(-\frac{p^l\eta^{ij}}{(x_1-x_2)^2|x_3-z|^2}+4\frac{p^i\eta^{jl}k_2k_3}{(x_2-x_3)^2|x_1-z|^2}\Bigg)\times\\
&\quad\times 2\Bigg[-\frac{\eta^{ij}}{(x_1-x_2)^2}\frac{\eta^{l\mu}k_3^\nu-\eta^{l\nu}k_3^\mu}{|x_3-z|^2}-\\
&\qquad-4\frac{\eta^{ij}}{x_1-x_2}\Bigg(-\frac{\eta^{l\mu}}{x_3-z}\Bigg(\frac{k_3^\nu(k_1k_2)}{(x_1-x_2)(x_3-\bar z)}-\frac{k_2^\nu(k_1k_3)}{(x_1-x_3)(x_2-\bar z)}+\frac{k_1^\nu(k_3k_2)}{(x_2-x_3)(x_1-\bar z)}\Bigg)+\\
&\qquad\qquad+\frac{\eta^{l\nu}}{x_3-\bar z}\Bigg(\frac{k_3^\mu(k_1k_2)}{(x_1-x_2)(x_3-z)}-\frac{k_2^\mu(k_1k_3)}{(x_1-x_3)(x_2-z)}+\frac{k_1^\mu(k_1k_3)}{(x_2-x_3)(x_1-z)}\Bigg)\Bigg)+\\
&\qquad+4\frac{p^i(z-\bar z)}{(x_2-x_3)|x_1-z|^2}\Bigg(\eta^{jl}\Bigg(\frac{k_3^\mu-k_2^\nu}{(x_3-z)(x_2-\bar z)}-\frac{k_2^\mu-k_3^\nu}{(x_2-z)(x_3-\bar z)}\Bigg)+\\
&\qquad\qquad+(k_2k_3)\Bigg(\frac{\eta^{l\mu}\eta^{j\nu}}{(x_3-z)(x_2-\bar z)}-\frac{\eta^{l\nu}\eta^{j\mu}}{(x_2-z)(x_3-\bar z)}\Bigg)\Bigg)\Bigg]-\\
&\quad-\Bigg(2p^ip^j(\eta^{l\mu}k_3^\nu-\eta^{l\nu}k_3^\mu)-\frac{1}{3}p^ip^jp^l\eta^{\mu\nu}\Bigg)\frac{(z-\bar z)^2}{|x_1-z|^2|x_2-z|^2|x_3-z|^2}+\\
&+(\text{cyclic permutation of } i,j,l)\Bigg\}\times\\
&\times|x_1-x_2|^{4k_1k_2}|x_1-x_3|^{4k_1k_3}|x_2-x_3|^{4k_2k_3}(x_1-z)^{2k_1p}(x_1-\bar z)^{2k_1Dp}(x_2-z)^{2k_2p}\times\\
&\times(x_2-\bar z)^{2k_2Dp}(x_3-z)^{2k_3p}(x_3-\bar z)^{2k_3Dp}(z-\bar z)^{pDp}C^{\text{ghost}}_{D_2}\frac{(x_1-x_2)(x_1-x_3)(x_2-x_3)}{z-\bar z}.
\end{aligned}
\tag{3.6}
$$

We choose to fix the open string operator coordinates at $x_1=\infty$, $x_2=1$, $x_3=0$. To simplify the final expression we use the symmetry of the graviton polarization tensor. The expression to be integrated over the closed string position on the world-sheet is:

$$
\begin{aligned}
A\sim\int_{\mathcal{H}^+}d^2z\Bigg\{&-u\,\mathrm{Tr}(\epsilon D)(\zeta_1p)(\zeta_2\zeta_3)+\mathrm{Tr}(\epsilon D)(-(\zeta_1\zeta_2)(\zeta_3p)-\\
&-4(s+1)(\zeta_1\zeta_2)(\zeta_3\epsilon k_3))\frac{1}{|z|^2}+4t\frac{(\zeta_1\zeta_2)(\zeta_3\epsilon k_2)}{|1-z|^2}\times\\
&\times(-4t(\zeta_1\zeta_2)(\zeta_3\epsilon k_2)+4u(\zeta_1\zeta_2)(\zeta_3\epsilon k_1))\frac{(\frac{z+\bar z}{2})}{|z|^2|1-z|^2}+\\
&+(\zeta_1p)\Bigg(16(\zeta_2\zeta_3)(k_2\epsilon k_3)+4u(\zeta_2\epsilon\zeta_3)-\frac{1}{12}(\zeta_2p)(\zeta_3p)-\\
&\qquad-16(\zeta_2p)(\zeta_3\epsilon k_3)\Bigg)\frac{(\frac{z-\bar z}{2})^2}{|z|^2|1-z|^2}+\\
&+(\text{cyclic permutation of }(k_I,\zeta_I))\Bigg\}\times
\end{aligned}
$$

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$$\times |1-z|^{s+u}|z|^{t+u}(z-\bar{z})^{-s-t-u-1} + (k_2,\zeta_2) \leftrightarrow (k_3,\zeta_3)\,. \tag{3.7}$$

We use formula (A.8) to perform the integrals. Permuting the scalars allows us to combine several terms together and eliminate undesired poles for $s, t, u = -1$ which appear in the amplitude. So the string amplitude takes the final form:

$$A = \mathcal{N}\frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)\Gamma(-\frac{s+t+u}{2})}{\Gamma(1-\frac{t+u}{2})\Gamma(1-\frac{s+u}{2})\Gamma(1-\frac{t+s}{2})\Gamma(1-\frac{s}{2})\Gamma(1-\frac{t}{2})\Gamma(1-\frac{u}{2})}F(1,2,3,\epsilon)\,, \tag{3.8}$$

where $\mathcal{N}$ is a normalization constant to be determined and $F(1,2,3,\epsilon)$ is a form factor depending on the polarizations and momenta:

$$\begin{aligned} F(1,2,3,\epsilon) = {} & (s+u)(s+t)\left(\alpha_1 + \frac{\alpha_2}{2}\right) + t(s+t)\alpha_3 + u(s+u)\alpha_4 + \\ & + (s+t+u)(s\alpha_5 + \alpha_6) + (\text{cyclic permutation of } (k_I,\zeta_I))\,. \end{aligned} \tag{3.9}$$

The coefficients α_i are given by:

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(\zeta_1\zeta_2)(\zeta_3 p)\,\mathrm{Tr}(\epsilon D)\,, \\ \alpha_2 &= 2(\zeta_1\zeta_2)(\zeta_3\epsilon k_3)\,, \\ \alpha_3 &= -2(\zeta_1\zeta_2)(\zeta_3\epsilon k_2)\,, \\ \alpha_4 &= -2(\zeta_1\zeta_2)(\zeta_3\epsilon k_1)\,, \\ \alpha_5 &= 2(\zeta_3 p)(\zeta_1\epsilon\zeta_2)\,, \\ \alpha_6 &= 8(\zeta_3 p)(\zeta_1\zeta_2)(k_1\epsilon k_2) - 8(\zeta_1 p)(\zeta_2 p)(\zeta_3\epsilon k_3) - \frac{2}{3}(\zeta_1 p)(\zeta_2 p)(\zeta_3 p)\,\mathrm{Tr}(\epsilon D)\,, \end{aligned} \tag{3.10}$$

This amplitude contains poles which can be understood by factorizing the world sheet as in (figure 1). The poles appear when open string vertex operators collide on the boundary of the disc. When this happens, open strings can propagate as intermediate states between the two world sheets in (figure 1).

4. Comparison with Born-Infeld action

In order to determine the (α') corrections to the Born-Infeld action (4.1) we will have to find the normalization constant $\mathcal{N}$ of our amplitude. We will expand the string amplitude for small momenta and compare the leading contribution with the field theory scattering amplitude from the Born-Infeld action. This will provide with a check of our calculation as well. We follow the methods and conventions of [21]. The world-volume theory of the D-brane includes the massless fields $X^\mu(\sigma)$ and $A^\alpha(\sigma)$. The fields $X^\mu(\sigma)$, where σ^α the world-volume coordinates, are the embedding of the p-brane in the ambient space-time. The fields $A^\alpha(\sigma)$ are the gauge fields of the U(1) abelian gauge theory on the brane. In this section we only consider the abelian case which corresponds to a single brane dynamics. In addition there are supersymmetric

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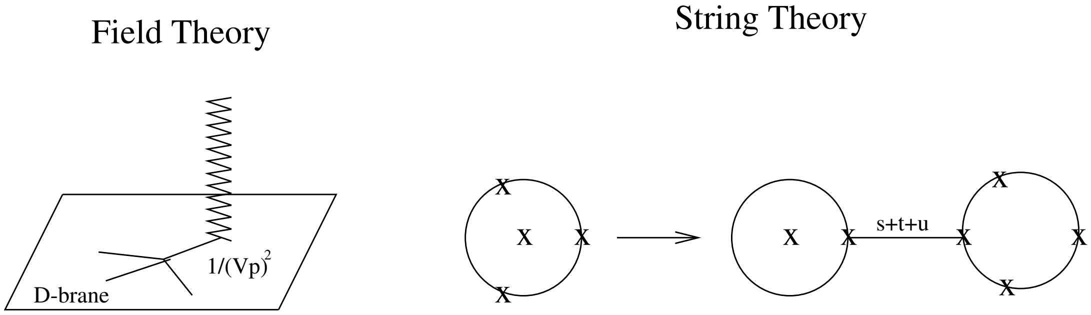

Figure 1: Factorization of the world-sheet giving rise to poles from the exchange of open string modes between the two world-sheets. The equivalent field theory diagram for small momenta (massless pole for $(Vp)^2 = 0$). The wavy lines represent gravitons and the plain ones open strings.

partners of those fields but they are irrelevant to our case. The low energy dynamics of the brane are encoded in the DBI-WZ action:

$$\begin{aligned} S_{D_p} = T_{(p)} \int dx^{p+1} e^{-\phi} \sqrt{\det\left((G_{\mu\nu} + B_{\mu\nu})\partial_\alpha X^\mu \partial_\beta X^\nu + 2\pi\alpha' F_{\alpha\beta}\right)} - \\ - iT_{(p)} \int_{p+1} \exp(2\pi\alpha' F + B) \wedge \sum_q C_q \,, \end{aligned} \tag{4.1}$$

where $F_{\alpha\beta}$ the gauge boson field strength and the q-form potentials C_q with q odd for type-IIA and even for type-IIB are the Ramond-Ramond background fields.

In the first step we will expand the Born-Infeld action to get the contact terms involving three scalars and one graviton. We work in the Einstein frame with metric $G_{\mu\nu} = e^{\Phi/2} g_{\mu\nu}$. This is necessary because in this frame the bulk action for the graviton takes the Hilbert-Einstein form $(\sqrt{-g})^{1/2} \frac{\mathbf{R}}{2\kappa^2}$ where $\mathbf{R}$ the bulk Ricci tensor and κ the gravitational constant. In this frame the DBI action takes the form

$$S_{DBI} = T_p \int d^{p+1}\sigma \,\mathrm{Tr}\left(e^{\frac{p-3}{4}\Phi} \sqrt{-\det(\tilde{g}_{\alpha\beta} + e^{-\frac{\Phi}{2}} \tilde{B}_{\alpha\beta} + 2\pi l_s^2 e^{-\frac{\Phi}{2}} F_{\alpha\beta})} \right) , \tag{4.2}$$

where $\tilde{g}_{\alpha\beta}$, $\tilde{B}_{\alpha\beta}$ are the pull-backs on the brane of the corresponding bulk tensors

$$\tilde{g}_{\alpha\beta} = g_{\mu\nu} \partial_\alpha X^\mu \partial_\alpha X^\mu \tag{4.3}$$

and similar expression for $\tilde{B}_{\alpha\beta}$.

We work in the static gauge where the position of the p-brane is fixed in the transverse dimensions. The world-volume coordinates coincide with the bulk coordinates x^μ for $\mu = 0, \ldots, p$ (see appendix B) which implies

$$\begin{aligned} \partial_\alpha X^\beta &= \delta_\alpha^\beta \\ g_{\mu\nu}(X) &= g_{\mu\nu}(\sigma, X^i) \,, \end{aligned} \tag{4.4}$$

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with σ the world-volume coordinates. The fields X^i describe transverse fluctuations of the brane. In this gauge (4.3) takes the form

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + 2g_{i(\alpha}\partial_{\beta)}X^i + g_{ij}\partial_\alpha X^i \partial_\beta X^j \,. \tag{4.5}$$

We want to expand for fluctuations around a flat empty space i.e. $g_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = \Phi = 0$, therefore we have

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + 2h_{\mu\nu}(\sigma, X^i) \\ B_{\mu\nu} &= -2b_{\mu\nu}(\sigma, X^i) \\ \Phi &= \sqrt{2}\phi \,, \end{aligned} \tag{4.6}$$

where we have set $\kappa = 1$. Applying (4.5) and (4.6) the Born-Infeld action takes a simple form and we are able to expand the square root of the determinant using the formula

$$\sqrt{\det(\delta^\alpha_{\ \beta} + M^\alpha_{\ \beta})} = 1 + \frac{1}{2}M^\alpha_{\ \alpha} - \frac{1}{4}M^\alpha_{\ \beta}M^\beta_{\ \alpha} + \frac{1}{8}(M^\alpha_{\ \alpha})^2 + \cdots \,. \tag{4.7}$$

The three scalars and one graviton couplings from the DBI are

$$\begin{aligned} L_{\text{contact}} = T_p \Big\{ &-(\partial_\alpha X^i \partial^\beta X_i \partial^\alpha X^j h_{j\beta} + \partial^\alpha X^i \partial^\beta X_i \partial_\beta X^j h_{j\alpha}) + \\ &+ \partial_\alpha X^i \partial^\alpha X_i \partial^\beta X^j h_{j\beta} \frac{1}{6} X^i X^j X^k \partial_i \partial_j \partial_k h^\alpha_{\ \alpha} + X^j X^k \partial_j \partial_k h_{i\alpha} \partial^\alpha X^i + \\ &+ \frac{1}{2} X^i (\partial_i h^\alpha_{\ \alpha})(\partial X)^2 - X^j(\partial_j h^{\alpha\beta})\partial_\alpha X^i \partial_\beta X_i + X^k(\partial_k h^{ij})\partial_\alpha X_i \partial^\alpha X_i \Big\}, \end{aligned} \tag{4.8}$$

where the first three terms come from direct expansion of the square root and the remaining from Taylor expansion of the graviton in the transverse space, in contact terms with zero, one and two scalars respectively. i.e.

$$\begin{aligned} \frac{1}{2}M^\alpha_{\ \alpha} &= h^\alpha_{\ \alpha}(\sigma, X) + \cdots \\ &= h^\alpha_{\ \alpha}(\sigma, X)|_{X=0} + X^i \partial_i h^\alpha_{\ \alpha}(\sigma, X)|_{X=0} + \frac{1}{2} X^i X^j \partial_i \partial_j h^\alpha_{\ \alpha}(\sigma, X)|_{X=0} + \\ &\quad + \frac{1}{6} X^i X^j X^k \partial_i \partial_j \partial_k h^\alpha_{\ \alpha}(\sigma, X)|_{X=0} + \cdots \,. \end{aligned} \tag{4.9}$$

Using for the scalar and graviton fluctuations the following functional form

$$h_{\mu\nu}(\sigma, X^i) = \epsilon_{\mu\nu} e^{i(p^\alpha \sigma_\alpha + p^i X_i)}, \qquad X^i(\sigma) = \zeta^i e^{ik\cdot\sigma} \tag{4.10}$$

we can write down the Feynman rules for the vertices.

The polarizations $\epsilon_{\mu\nu}$ and ζ^i satisfy the same conditions as in the string amplitude (3.4) in section 3. The contribution of the interactions (4.8) to the one graviton

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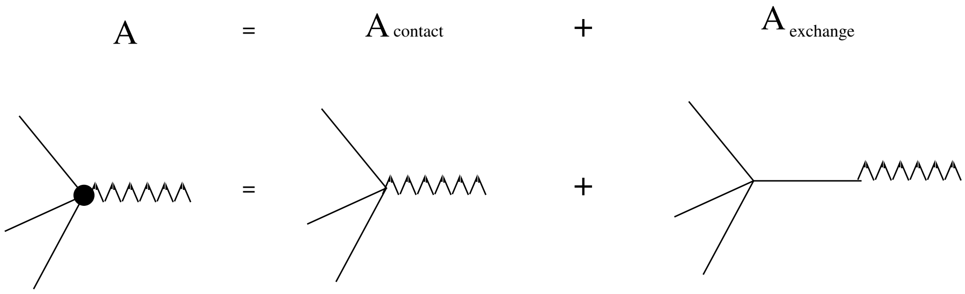

Figure 2: The field theory amplitude A is the sum of a contact part, A^{contact} and a scalar exchange part, A^{exchange}.

with three scalars amplitude is

$$
\begin{aligned}
A^{\text{contact}} &= T_p\Big\{ -\frac{(\zeta_1\zeta_2)}{2}(s(\zeta_3\epsilon k_3) - t(\zeta_3\epsilon k_2) - u(\zeta_3\epsilon k_1)) + \frac{1}{3}(\zeta_1 p)(\zeta_2 p)(\zeta_3 p)\,\mathrm{Tr}(\epsilon V) + \\
&\quad + 2(\zeta_1 p)(\zeta_2 p)(\zeta_3\epsilon k_3) - \frac{s}{4}(\zeta_3 p)(\zeta_1\zeta_2)\,\mathrm{Tr}(\epsilon V) - 2(\zeta_3 p)(\zeta_1\zeta_2)(k_1\epsilon k_2) - \\
&\quad - \frac{s}{2}(\zeta_3 p)(\zeta_1\epsilon\zeta_2) + (\text{cyclic permutation of } (k_I,\zeta_I))\Big\}\,. \qquad (4.11)
\end{aligned}
$$

These terms alone do not reproduce the string amplitude. As we can see from the full expression in section 3, the amplitude contains a pole (see figure 1) of the form $1/q^2$, where $q^2 = -\frac{s+t+u}{2}$, see (3.2). This pole is due to the exchange of a virtual scalar particle between the graviton and the three open strings, (see figure 2). To construct these exchange terms we need the graviton-scalar mixing and four scalar vertices from the DBI.

$$
\begin{aligned}
L_{(X^4)} &= T_p\left(-\frac{1}{4}\partial_\alpha X^i\partial_\beta X_i\partial^\alpha X^j\partial^\beta X_j + \frac{1}{8}\partial_\alpha X^i\partial^\alpha X_i\partial^\beta X_j\partial_\beta X^j\right) \\
L_{(Xg)} &= T_p(X^i\partial_i h^\alpha_\alpha + 2\partial^\alpha X^i h_{\alpha i})\,. \qquad (4.12)
\end{aligned}
$$

These vertices are shown in (figure 3) along with the exchange diagram which contributes to our scattering amplitude. The field theory exchange amplitude can be written

$$
A^{\text{exchange}}_{q^2} = V^i_{(X^4)}P^j_i V^{(Xg)}_j\,, \qquad (4.13)
$$

where $P^j_i = \frac{i\delta^j_i}{q^2}\frac{1}{T_p}$ is the propagator of a scalar field with momentum q and the two vertices are the four scalar and scalar/graviton mixing vertices, respectively, from the DBI terms above. Notice that unlike [21] we have not absorbed T_p in our normalization of the scalars (see (4.10)), which accounts for the appearance of T_p in the scalar propagator. The final expression for $A^{\text{exchange}}_{q^2}$ is

$$
A^{\text{exchange}}_{q^2} = \frac{T_p(\zeta_1\zeta_2)tu}{8q^2}\{(\zeta_3 p)\,\mathrm{Tr}(\epsilon V) + 2(\zeta_3\epsilon(k_1+k_2+k_3))\}\,. \qquad (4.14)
$$

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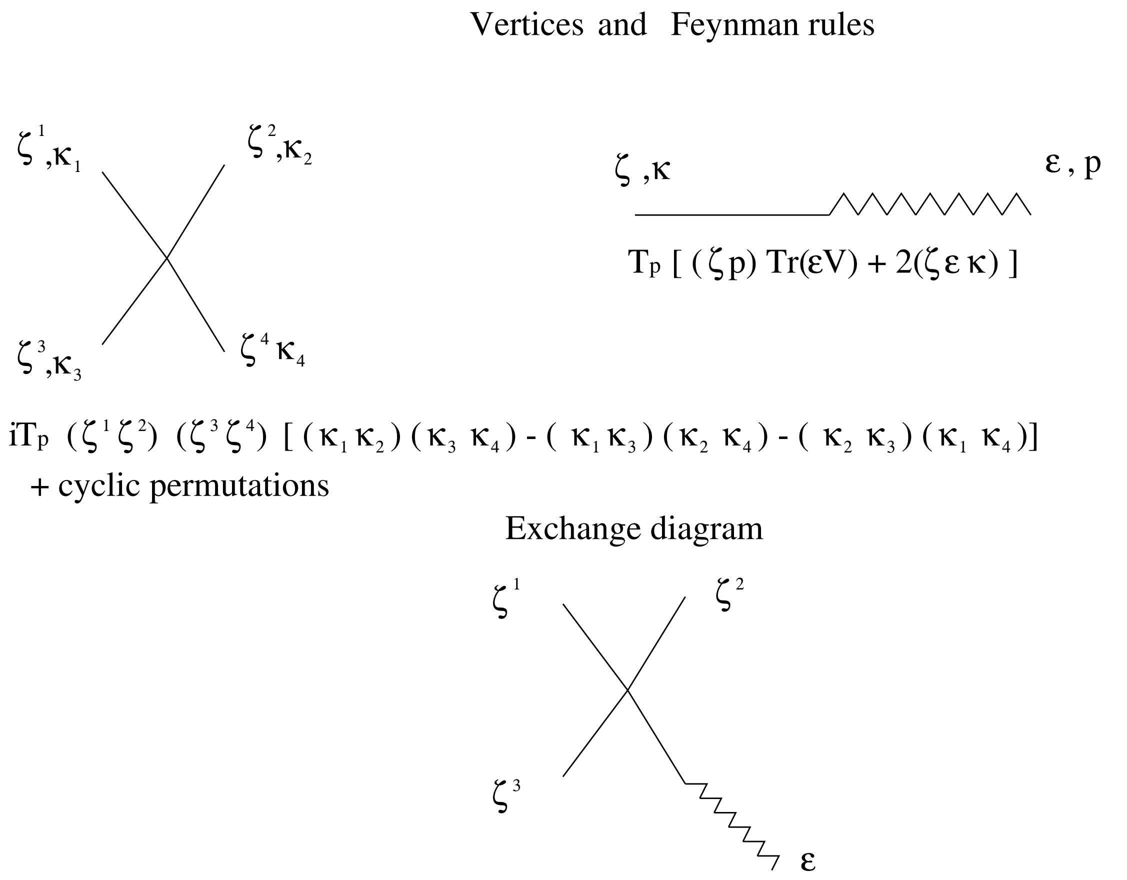

Figure 3: Feynman rules for the field theory calculation and the exchange diagram for the 3scalar+1graviton scattering process.

The final amplitude is $A_{FT} = A^{\text{contact}} + A_{q^2}^{\text{exchange}}$ (see figure 2). In order to compare the field theory amplitude with the string amplitude, we have to take the $\alpha' \to 0$ limit of the string amplitude. We restore α' in our expression (3.8) and we get

$$\begin{aligned} A &= \mathcal{N} \frac{\Gamma(1-\frac{\alpha' s}{2})\Gamma(1-\frac{\alpha' t}{2})\Gamma(1-\frac{\alpha' u}{2})\Gamma(-\frac{\alpha'}{4}s+t+u)}{\Gamma(1-\frac{\alpha'}{2}t+u)\Gamma(1-\frac{\alpha'}{4}s+u)\Gamma(1-\frac{\alpha'}{4}t+s)\Gamma(1-\frac{\alpha' s}{4})\Gamma(1-\frac{\alpha' t}{4})\Gamma(1-\frac{\alpha' u}{4})} \times \\ &\quad \times F(1,2,3,\epsilon) \\ &= \mathcal{N}\left[\frac{-4}{\alpha'(s+t+u)} + \mathcal{O}(\alpha')\right] \cdot F(1,2,3,\epsilon)\,. \end{aligned} \tag{4.15}$$

Direct comparison of (4.15) to A_{FT} verifies that our result is correct and fixes the normalization constant

$$\mathcal{N} = \frac{\alpha' T_p}{16}\,. \tag{4.16}$$

5. Derivative corrections to D-brane action I

The $(\mathcal{O}(\alpha'^2))$ derivative terms involve squares of pull-backs to the normal and tan-

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gent bundle of the Riemann tensor $R_{\mu\nu\rho\sigma}$. Their form is constrained by space-time and world-volume reparametrization invariance. There are also terms which involve the second fundamental form $\Omega^{\mu}_{\alpha\beta}$ of the hyperplane. The explicit form of these invariants as well as the definition of the second fundamental form are given in appendix B. In [18] the ($\mathcal{O}(\alpha'^2)$) corrections to the Born-Infeld action were determined by expanding the two closed, one closed with two open and four open string amplitudes to the subleading order in the momenta and comparing these with the set of allowed derivative terms of the form R^2, $R\Omega^2$ and Ω^4 respectively. The comparison with the 2 closed strings amplitude determines R^2 corrections to the DBI modulo terms involving the bulk Ricci tensor $R_{\mu\nu}$. The reason terms proportional to $R_{\mu\nu}$ cannot be determined, is that this tensor vanishes when we impose the lowest order (in α') equations of motion for the graviton (B.8). There is only one ambiguity left for the R^2 terms, which is proportional to the Gauss-Bonnet term

$$L_{GB} = \frac{\sqrt{\tilde{g}}}{32\pi^2}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta} + \hat{R}^2\right), \tag{5.1}$$

where the Riemann tensors are the pull-backs to the world-volume of the corresponding bulk tensors and $\hat{R}_{\alpha\beta}$, $\hat{R}$ are obtained by contracting world-volume indices only (see appendix B). The Gauss-Bonnet term is a total derivative at linear expansion of the Riemann tensors as can be explicitly checked. This ambiguous coefficient was fixed in [18] by using IIA/F-theory duality. It was found that it is vanishing. Nevertheless we will keep the coefficient of this term as undetermined in our present analysis. As we shall shortly explain there are additional ambiguities in the proposed lagrangian. Through our amplitude calculations we will be able to relate the coefficients of these ambiguous terms among themselves. We will consequently use the conjectured duality to fix the values of all these coefficients.

There are also higher derivative terms involving the second fundamental form Ω of the hyperplane. Since the second fundamental form is a scalar field excitation to linear order, we can use the one closed with two open and four open string amplitudes to determine the coefficients of $R\Omega^2$ and Ω^4 corrections to the lagrangian. In [18] it was found that the string amplitudes mentioned above are reproduced by the lagrangian

$$\begin{aligned} L^{(p)} = T_p\, e^{-\phi}\sqrt{\tilde{g}}\Bigg[1 - \frac{1}{24}\frac{(4\pi^2\alpha')^2}{32\pi^2}\Bigg(&(R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta} - 2(R_T)_{\alpha\beta}(R_T)^{\alpha\beta} - \\ &- (R_N)_{\alpha\beta ij}(R_N)^{\alpha\beta ij} + 2\bar{R}_{ij}\bar{R}^{ij}\Bigg)\Bigg]. \end{aligned} \tag{5.2}$$

The tensors R_T and R_N are constructed from the world-volume and normal bundle connections (see appendix B). In analogy with the R^2 terms, invariants involving the trace of the second fundamental form, which to linear order is the equation of motion for the scalar field (B.8), vanish and therefore cannot be determined by comparison with the above mentioned amplitudes.

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Beyond these ambiguities we have again combinations of the $R\Omega^2$ and Ω^4 which turn out to be total derivatives for linear expansion in the fields of the tensors. In the following formula we summarize the results found by [18] and the ambiguities involved. As explained in appendix B we have written their expression in a slightly different form using (B.6) and (B.7). We have also included terms proportional to the trace of Ω. These terms are non-vanishing when we expand the second fundamental form beyond the leading order to get a graviton field.

$$
\begin{aligned}
L^{(p)} = T_p e^{-\phi}\sqrt{\tilde{g}}\Bigg[1 - \frac{1}{24}\frac{(4\pi^2\alpha')^2}{32\pi^2}\Bigg(&(R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R^{\alpha\beta} - 4R_{\alpha\beta}(\Omega^{\alpha\gamma}\cdot\Omega_{\gamma}{}^{\beta}) - \\
&- 2(\Omega_{\alpha\gamma}\cdot\Omega^{\alpha\delta})(\Omega^{\beta\gamma}\cdot\Omega_{\beta\delta}) - (R_N)_{\alpha\beta ij}(R_N)^{\alpha\beta ij} + \\
&+ 2\bar{R}_{ij}\bar{R}^{ij} + L^{\mathrm{ambig}}\Bigg) + \mathcal{O}(\alpha'^4)\Bigg],
\end{aligned}
\tag{5.3}
$$

where $\bar{R}_{ij} \equiv \hat{R}_{ij} + g^{\alpha\alpha'}g^{\beta\beta'}\Omega_{i|\alpha\beta}\Omega_{j|\alpha'\beta'}$ defined in [18] and we have included all the ambiguities in the last term of the above expression. Using (B.16), (B.17) this lagrangian term takes the form

$$
\begin{aligned}
L^{\mathrm{ambig}} = \Bigg[&c_1(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta} + \hat{R}^2) + \\
&+ c_2\Big(4R_{\alpha\beta\gamma\delta}(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta}) - 8\hat{R}_{\alpha\beta}(\Omega^{\alpha\gamma}\cdot\Omega_{\gamma}{}^{\beta}) + 2\hat{R}(\Omega^{\alpha\beta}\cdot\Omega_{\alpha\beta})\Big) + \\
&+ c_3\Big(2(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta}) - 2(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})(\Omega^{\alpha\delta}\cdot\Omega^{\beta\gamma}) - \\
&\quad - 4(\Omega_{\alpha\gamma}\cdot\Omega_{\beta}{}^{\gamma})(\Omega^{\alpha\gamma}\cdot\Omega_{\gamma}{}^{\beta}) + (\Omega^{\alpha\beta}\cdot\Omega_{\alpha\beta})(\Omega^{\gamma\delta}\cdot\Omega_{\gamma\delta})\Big) + \\
&+ c_4\Big((\Omega_{\gamma}{}^{\gamma}\cdot\Omega_{\alpha}{}^{\beta})(\Omega_{\delta}{}^{\alpha}\cdot\Omega_{\beta}{}^{\delta})\Big) + c_5\Big(\hat{R}_{\alpha\beta}(\Omega^{\alpha\beta}\cdot\Omega_{\gamma}{}^{\gamma})\Big)\Bigg].
\end{aligned}
\tag{5.4}
$$

These are all the ambiguities, involving one or no trace of the second fundamental form.

In [18] it was guessed that the $R\Omega^2$ terms in the second line of $L^{(p)}$ can be written in terms of invariants constructed from $(R_T)_{\alpha\beta}$ as in (5.2). Using the Gauss-Codazzi equations (B.6) and (B.7) and the definition $(R_T)_{\alpha\beta} = (R_T)^{\gamma}{}_{\alpha\gamma\beta}$ these terms become

$$
\begin{aligned}
(R_T)_{\alpha\beta}(R_T)^{\alpha\beta} = &R_{\alpha\beta}R^{\alpha\beta} + 2R_{\alpha\beta}(\Omega^{\alpha\gamma}\cdot\Omega_{\gamma}{}^{\beta}) + (\Omega_{\alpha\gamma}\cdot\Omega^{\alpha\delta})(\Omega^{\beta\gamma}\cdot\Omega_{\beta\delta}) - \\
&- 2(\Omega_{\gamma}{}^{\gamma}\cdot\Omega_{\alpha}{}^{\beta})(\Omega_{\delta}{}^{\alpha}\cdot\Omega_{\beta}{}^{\delta}) - 2\hat{R}_{\alpha\beta}(\Omega^{\alpha\beta}\cdot\Omega_{\gamma}{}^{\gamma}) + (\Omega_{\gamma}{}^{\gamma}\cdot\Omega_{\alpha}{}^{\beta})(\Omega_{\gamma}{}^{\gamma}\cdot\Omega_{\alpha}{}^{\beta}).
\end{aligned}
\tag{5.5}
$$

It is obvious from the equation above that if this were the case then the terms proportional to c_4 and c_5 should appear. We will see that these terms cannot be determined through our string amplitude methods and therefore remain ambiguous in the effective lagrangian. In addition comparison with our string amplitudes will show that the ambiguous terms combine to form the Gauss-Bonnet lagrangian for the induced metric $g_{\alpha\beta}$ up to terms proportional to the trace of the second fundamental form.

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Now we proceed with the Field Theory computation. Contact terms involving one closed and three open strings come from pull-backs and Taylor expansion in the graviton of the $R\Omega^2$ terms, as well as from Ω^4 terms from expansion of the second fundamental form beyond the leading order. It is obvious that the coefficients c_1 and c_5 cannot be determined by employing the three scalars and one graviton string amplitude. Both terms involve at least two gravitons and we will need the two graviton with one scalar amplitude to fix them. As an example of the expansion needed for the computation we write down the pull-back of one $R\Omega^2$ term keeping only the pieces relevant to our case

$$R^{\alpha\beta}{}_{ij}\Omega^i_{\alpha\gamma}\Omega^{j\gamma}_{\beta} \to \partial_\alpha X_\mu R^{\mu\beta}_{ij}\Omega^i_{\alpha\gamma}\Omega^{j\gamma}_{\beta} + \xi^\mu_i R^{\alpha\beta}{}_{\mu j}\Omega^i_{\alpha\gamma}\Omega^{j\gamma}_{\beta} + (\alpha \leftrightarrow \beta, i \leftrightarrow j)\,. \tag{5.6}$$

In addition the second fundamental form in static gauge has the following expansion

$$\Omega^i_{\alpha\beta} = \partial_\alpha\partial_\beta X^i + \Gamma^i_{\alpha\beta} + \cdots \text{ 2-particle terms}\,. \tag{5.7}$$

The expansion of the Christoffel symbol around a flat background gives to lowest order one graviton field.

$$\Gamma^i_{\alpha\beta} = [\partial_\beta h_{\alpha i} + \partial_\alpha h_{\beta i} - \partial_i h_{\alpha\beta}]\,. \tag{5.8}$$

This expansion can be used for one of the second fundamental forms in Ω^4 to give contact interactions of three scalars and one graviton.

A tedious calculation gives the contribution of the contact terms to the field theory amplitude

$$\begin{aligned} A^{\text{contact}}_{FT} = -\frac{T_p(4\pi^2\alpha')^2}{24(32\pi^2)}\Bigg\{ & (s^3+u^2s+t^2s)\frac{\beta_1}{4} + (s^2+t^2+u^2)\left(-\frac{\beta_2}{6} - \beta_3 + \frac{\beta_4}{8}\right) + \\ & + (s(s^2+t^2+u^2) - ut(s-u-t))\frac{\beta_5}{8} - \\ & - (s^2u + stu + u^3 - u^2t)\frac{\beta_6}{4} - (s^2t + stu + t^3 - t^2u)\frac{\beta_7}{4} - \\ & - (s^2(t+u) + 4stu - 2s^3)\frac{\beta_8}{8} + (c_2 - c_3)\left[u^2\frac{\beta_9}{2} + t^2\frac{\beta_{10}}{2} - \right. \\ & \left. - ut\beta_4 - us\beta_{11} - ts\beta_{12} + s^2\frac{\beta_{13}}{2}\right] \times \\ & \times sut\left[\left(-\frac{c_2}{4} + \frac{c_4}{32}\right)\beta_5 + \left(-\frac{c_3}{2} + \frac{c_4}{16}\right)(\beta_6 + \beta_7 + \beta_8)\right] + \\ & + (\text{cyclic permutation of } (k_I, \zeta_I))\Bigg\}\,. \end{aligned} \tag{5.9}$$

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The coefficients β_i are functions of the polarizations and momenta

$$
\begin{aligned}
\beta_1 &= (\zeta_1\epsilon\zeta_2)(\zeta_3 p)\,, & \beta_2 &= (\zeta_1 p)(\zeta_2 p)(\zeta_3 p)\,\mathrm{Tr}(\epsilon V)\,,\\
\beta_3 &= (\zeta_1 p)(\zeta_2 p)(\zeta_3\epsilon k_3)\,, & \beta_4 &= (\zeta_1\zeta_2)(\zeta_3 p)(k_1\epsilon k_2)\,,\\
\beta_5 &= (\zeta_1\zeta_2)(\zeta_3 p)\,\mathrm{Tr}(\epsilon V)\,, & \beta_6 &= (\zeta_1\zeta_2)(\zeta_3\epsilon k_3)\,,\\
\beta_7 &= (\zeta_1\zeta_2)(\zeta_3\epsilon k_1)\,, & \beta_8 &= (\zeta_1\zeta_2)(\zeta_3\epsilon k_2)\,,\\
\beta_9 &= (\zeta_1\zeta_2)(\zeta_3 p)(k_1\epsilon k_1)\,, & \beta_{10} &= (\zeta_1\zeta_2)(\zeta_3 p)(k_2\epsilon k_2)\,,\\
\beta_{11} &= (\zeta_1\zeta_2)(\zeta_3 p)(k_1\epsilon k_3)\,, & \beta_{12} &= (\zeta_1\zeta_2)(\zeta_3 p)(k_2\epsilon k_3)\,,\\
\beta_{13} &= (\zeta_1\zeta_2)(\zeta_3 p)(k_3\epsilon k_3)\,, & & \qquad\qquad (5.10)
\end{aligned}
$$

$A_{FT}^{\mathrm{contact}}$ does not reproduce the full string amplitude. The previously computed amplitudes [20, 18] had no poles to the subleading order in the momenta because the trilinear bulk and scalar-graviton mixing vertices are presumably protected by supersymmetry and do not receive derivative corrections. Nevertheless, as already mentioned above, the four scalars vertex has derivative corrections, so one needs to include scalar exchange diagrams of these vertices with the scalar-graviton mixing vertex. This is in exact analogy with the DBI case of section 4. Each Ω^4 and $\Omega^3\,\mathrm{Tr}\,\Omega$ term (with $\mathrm{Tr}\,\Omega$ giving the off-shell exchanged scalar) contributes a four scalars vertex ($\sim\ k^8$) where k the momentum of the scalars. The mixing diagrams formed between these vertices and $L_{(Xg)}$ from (4.12) are of the same order as the contact terms. The explicit form of the exchange diagram contribution is given by

$$
\begin{aligned}
A_{FT}^{\mathrm{exchange}} = -\frac{T_p(4\pi^2\alpha')^2}{24(32\pi^2)}\Bigg\{\left[\frac{(s^2-ut)ut}{4(s+u+t)}+sut\left(\frac{c_3}{4}-\frac{c_4}{32}\right)\right](\beta_5+2\beta_6+2\beta_7+2\beta_8)+\\
+\ (\text{cyclic permutation of }(k_I,\zeta_I))\Bigg\}\,. \qquad (5.11)
\end{aligned}
$$

We now expand the three scalars and one graviton amplitude from (4.15) to subleading order in momenta

$$
A = -\frac{T_p}{8}\left[\frac{2}{s+t+u}+\frac{1}{24}\frac{(4\pi^2\alpha')^2}{32\pi^2}\frac{s^2+t^2+u^2}{s+t+u}+\mathcal{O}(\alpha'^4)\right]\cdot F(1,2,3,\epsilon)\,. \qquad (5.12)
$$

The subleading piece of the expansion, as expected, has a pole which is not canceled. The full field theory amplitude $A_{FT} = A_{FT}^{\mathrm{contact}} + A_{FT}^{\mathrm{exchange}}$ (figure 2) is compared to the string amplitude expansion. The two expressions come to complete agreement for the following values of the unknown coefficients

$$
c_2 = c_3\,, \qquad (5.13)
$$

while the exchange and contact contributions, from the lagrangian term 1 to c_4, cancel against each other. Therefore

$$
c_4 \to \text{undetermined}\,.
$$

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As we already saw the coefficients c_2, c_3 which correspond to terms that do not involve the trace of Ω cannot be completely fixed by our computations but they are proportional to each other. On the other hand the coefficient c_4 of the interaction term proportional to the trace of the second fundamental form is completely undetermined. We will explain the source of this remaining ambiguity in our conclusions.

In the next section we will try to determine the remaining coefficients c_1, c_5. The terms proportional to these coefficients are non-vanishing for a two graviton with one scalar amplitude. Therefore we will need to compute the string amplitude of two closed strings with one open string excitation.

6. Scalar-two graviton scattering amplitude

In this section we will determine the remaining ambiguities in $L^{(p)}$. The terms we are interested in are proportional to c_1, c_2 and c_5. The c_1 and c_5 terms have an expansion which contributes to amplitudes at least quadratic in the graviton field. The two graviton scattering amplitude is not enough to fix the c_1 term as explained in section 5. The next simplest case where such terms contribute involves two gravitons and one scalar excitation. In addition the amplitude receives contributions from the c_2 terms as well. In conclusion we need to compute the string amplitude with two closed and one open strings. Comparison of the low energy expansion of this amplitude with $L^{(p)}$ will eventually fix some of the ambiguous terms.

We could go on and compute the string amplitude using the correlators given in section 2 but it will prove difficult to perform the integration over the position of string vertex operators on the complex plane. On the other hand, we can use correlators on the disc rather than the half complex plane. The Green's functions on the disc are found using the method of image charges on a two dimensional surface [25]. Each string inserted at position z on the disc has an image at $\frac{1}{\bar{z}}$. Imposing Neumann or Dirichlet boundary conditions we find the correlators on the disc [24]:

$$\begin{aligned}
\langle X^\mu(z)X^\nu(w)\rangle &= -\eta^{\mu\nu} ln(z-w)\\
\langle X^\mu(z)\bar{X}^\nu(\bar{w})\rangle &= -D^{\mu\nu} ln(1-z\bar{w})\\
\langle \psi^\mu(z)\psi^\nu(w)\rangle &= -\frac{\eta^{\mu\nu}}{z-w}\\
\langle \psi^\mu(z)\bar{\psi}^\nu(\bar{w})\rangle &= +i\frac{D^{\mu\nu}}{1-z\bar{w}}\,.
\end{aligned} \tag{6.1}$$

From the form of the correlators we see that we can still use the substitutions (2.7) for right movers in terms of left movers with anti-holomorphic arguments. Holomorphic and anti-holomorphic operators have non-trivial correlators given by (6.1). For the superghost fields we find:

$$\langle e^{-\phi(z)}e^{-\phi(\bar{w})}\rangle = \frac{1}{1-z\bar{w}} \tag{6.2}$$

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and the ghost correlators can be determined from those of the sphere using the doubling trick and transformation properties of the fields under the involution $z' = \bar{z}^{-1}$ [5]:

$$
\begin{aligned}
\langle c(z_1)c(z_2)c(z_3)\rangle &= C^{\text{ghost}}_{D_2}(z_1-z_2)(z_2-z_3)(z_1-z_3)\\
\langle c(z_1)c(z_2)\bar{c}(\bar{z}_3)\rangle &= \left(\frac{\partial z_3'}{\partial \bar{z}_3}\right)^{-1}\langle c(z_1)c(z_2)c(z_3')\rangle\\
&= C^{\text{ghost}}_{D_2}(z_1-z_2)(1-z_1\bar{z}_3)(1-z_2\bar{z}_3)\,.
\end{aligned}
\tag{6.3}
$$

Before we proceed with the string computation we will give some useful formulae for computing the vertex operator correlators:

$$
\begin{aligned}
\langle \partial_z X^\mu(z) e^{i(Dp)\cdot X(\bar{w})}\rangle &= \frac{i(Dp)^\mu \bar{w}}{1-z\bar{w}}\\
\langle \partial_{\bar{z}} X^\mu(\bar{z}) e^{ip\cdot X(w)}\rangle &= \frac{i(Dp)^\mu w}{1-\bar{z}w}\\
\langle \partial_z X^\mu(z)\partial_{\bar{w}} X^\nu(\bar{w})\rangle &= \frac{\eta^{\mu\nu}}{(1-z\bar{w})^2}\,.
\end{aligned}
\tag{6.4}
$$

The two closed with one open string amplitude has the general form (2.10). We choose to fix the positions of one closed and one open string. We also put the fixed closed string vertex operator in the $(-1,-1)$ picture. The string amplitude takes the form:

$$
\begin{aligned}
A_{\text{string}} \simeq \int_{|z|\le 1} & d^2z\, \langle c(z')\bar{c}(\bar{z}')V^{\mu\nu}_{(-1,-1)}(z,\bar{z})\, V^{\rho\sigma}_{(0,0)}(z,\bar{z})c(x)V^i_{(0)}(x)\rangle\, \times\\
& \times (\epsilon D)_{\mu\nu}(\bar{\epsilon}D)_{\rho\sigma}\zeta_i\,,
\end{aligned}
\tag{6.5}
$$

where the vertex operators for the closed strings are given by the expressions (2.3) and for open by equation (2.9). The momenta for the closed strings with polarizations $\epsilon^{\mu\nu}$ and $\bar{\epsilon}^{\rho\sigma}$ are p^μ and $\bar{p}^\mu$ respectively. The momentum of the open string is k^μ.

We introduce kinematic invariants:

$$
\begin{aligned}
s = \bar{p}\cdot D\cdot p \qquad & t = \bar{p}\cdot p \qquad u = \bar{p}\cdot D\bar{p}\\
p\cdot D\cdot p = -2(s+t)-u \qquad & 2p\cdot k = -2\bar{p}\cdot k = s+t+u\,.
\end{aligned}
\tag{6.6}
$$

In the last two equations we used the conservation of momentum along the world-volume directions:

$$
(V\bar{p})^\mu + (Vp)^\mu + k^\mu = 0\,. \tag{6.7}
$$

The gauge transversality conditions are the same as for the three open with one closed string amplitude (3.4). The string amplitude in this case though, is far more more tedious than before. We can simplify the calculations involved by restricting the momenta and/or polarizations of one closed string to lie on the D-brane directions.

If we impose this restriction for both momenta and polarizations of the closed string $(\bar{\epsilon}^{\mu\nu}, \bar{p}^{\mu})$, then as it turns out, the amplitude does not encode enough information to fix the ambiquous terms. On the other hand restricting only the polarization $\bar{\epsilon}^{\mu\nu}$ on the world-volume is sufficient for our purpose. The gauge transversality conditions for $\bar{\epsilon}^{\mu\nu}$ become:

$$\begin{aligned} \bar{p}^{\mu}\bar{\epsilon}_{\mu\nu} &= \bar{p}^{\alpha}\bar{\epsilon}_{\alpha\beta} = 0 \\ \bar{\epsilon}^{\mu}_{\mu} &= \bar{\epsilon}^{\alpha}_{\alpha} = 0\,. \end{aligned} \tag{6.8}$$

In addition, the ambiguous terms proportional to c_1 involve pull-backs of the Riemann tensor with all the indices on the world-volume. Contact interaction terms with two gravitons and one scalar come from the pull-back of the Riemann tensor or the Taylor expansion of the graviton in the Riemann tensor:

$$R_{\alpha\beta\gamma\delta} \to X^i \partial_i R_{\alpha\beta\gamma\delta} + \partial_\alpha X^i R_{i\beta\gamma\delta} + \cdots\,,$$

where the terms omitted involve more than one scalars. In either case we have only one index of the bulk Riemann tensor on the normal directions. A similar discussion applies for the terms proportional to c_5. We can therefore focus our attention on terms involving only one normal index in the string amplitude. We need only to be careful in applying the gauge transversality conditions (3.4) to reduce any normal indices to tangent where this is possible. With these two restrictions the computation simplifies considerably and as we shall see shortly there is enough information encoded in the string amplitude to determine the ambiguous terms. The integrand of the string amplitude (6.5) takes the form:

$$\begin{aligned} &\langle :(\partial X^\alpha + i(\bar{p}\cdot\psi)\psi^\alpha)(z)e^{i\bar{p}\cdot X(z)}: :(\partial X^\beta + i((D\bar{p})\cdot\psi)\psi^\beta)(\bar{z})e^{i(D\bar{p})\cdot X(z)}: \times \\ &\times\ :c(x)(\partial X^i + 2i(k\cdot\psi)\psi^i)(x)e^{2ik\cdot X(x)}: \times \\ &\times :c(z')e^{-\phi(z')}\psi^\mu(z')e^{ip\cdot X(z')} :: c(\bar{z}')e^{-\phi(\bar{z}')}\psi^\mu(\bar{z}')e^{i(Dp)\cdot X(\bar{z}')}:\rangle\ \epsilon_{\alpha\beta}(\epsilon D)_{\mu\nu}\zeta_i\,. \end{aligned} \tag{6.9}$$

The integrand breaks into four different types of path integrals to be evaluated in the same manner as in section 3. We will fix the position of the $\epsilon_{\mu\nu}$ closed string at the center of the disc $z' = \bar{z}' = 0$ and of the open string at $x = 1$.

The final result after the evaluation of all path integrals is:

$$\begin{aligned} &\left\{ \frac{|1+z|^2}{|1-z|^2|z|^2}\left(-\operatorname{Tr}(\epsilon D)(p\bar{\epsilon}p)(\zeta p) + 4(\zeta\epsilon p)(p\bar{\epsilon}p) + 2(\zeta\epsilon\bar{\epsilon}p)(s+t+u)\right) \times \right. \\ &\quad \times \frac{1-|z|^2}{|1-z|^2|z|^2}\left(-\operatorname{Tr}(\epsilon D)(p\bar{\epsilon}p)(\zeta\bar{p}) + 4(\zeta\bar{p})(p\bar{\epsilon}\epsilon p) - (\zeta\bar{p})\operatorname{Tr}(\epsilon\bar{\epsilon})(2s+2t+u)\right) \times \\ &\quad \left. \times \frac{1}{|z|^2}(-u\operatorname{Tr}(\epsilon\bar{\epsilon})(\zeta p)) + \frac{z+\bar{z}}{|1-z|^2|z|^2}(-2u(\zeta\epsilon\bar{\epsilon}p)) \right\} \times \\ &\times|1-z|^{2(-u-t-s)}(1-|z|^2)^u|z|^{2t}\,. \end{aligned} \tag{6.10}$$

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The integration over the closed string coordinate z can be performed using the formulas from appendix C. The final result is:

$$A_{\text{string}} = \mathcal{N}\Big\{(2I_1+I_3)(-2\gamma_1+4\gamma_2+2(s+t+u)\gamma_3)+ \\ + I_2(-2\gamma_4+4\gamma_5-(2s+2t+u)\gamma_6)+I_3(-u\gamma_7)+I_1(-2u\gamma_3)\Big\}\,, \quad (6.11)$$

where the expressions I_1, I_2, I_3 are defined in appendix C and $\mathcal{N}$ a normalization constant. We have also defined:

$$\begin{aligned} \gamma_1 &= \mathrm{Tr}(\epsilon V)(p\bar{\epsilon}p)(\zeta p) & \gamma_2 &= (\zeta\epsilon V p)(p\bar{\epsilon}p) \\ \gamma_3 &= (\zeta\epsilon\bar{\epsilon}p) & \gamma_4 &= \mathrm{Tr}(\epsilon V)(p\bar{\epsilon}p)(\zeta\bar{p}) \\ \gamma_5 &= (\zeta\bar{p})(p\bar{\epsilon}\epsilon p) & \gamma_6 &= (\zeta\bar{p})\,\mathrm{Tr}(\epsilon\bar{\epsilon}) \\ \gamma_7 &= \mathrm{Tr}(\epsilon\bar{\epsilon})(\zeta p)\,. \end{aligned} \quad (6.12)$$

The final result involves generalized hypergeometric functions. Manipulating these functions to bring our result in a more compact form is a difficult task. In addition for comparison with the low energy effective action $L^{(p)}$ we should expand the hypergeometric functions for small s, t, u. There is not such known expansion of these functions. Fortunately we shall need neither the normalization constant $\mathcal{N}$ nor the small momenta expansion of (6.11). The relative coefficients of the various polarization contractions in the expressions (6.12) will be sufficient for determining the unknown coefficients c_1, c_5 as well as for verifying that our computation is indeed correct.

7. Derivative corrections to D-brane action II

Now we proceed with the field theory computation. Contact terms involving two gravitons and one scalar come from pull-backs and Taylor expansion of R^2 terms as well as from $R\Omega^2$ terms from expansion of the second fundamental form beyond the leading order. Extracting these contact terms is a very tedious process. As an example we give the contribution from one R^2 term:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \to \partial_\alpha X^i R_{i\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \partial_\alpha X^i R_{i\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \\ + \text{all other pull-backs } + X^i\partial_i(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})\,. \quad (7.1)$$

The contribution of this term is:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \to 2\Bigg[(\zeta p)\Bigg(\mathrm{Tr}(\epsilon\bar{\epsilon})\Bigg(-\frac{(s+t)(s+t+2u)}{4}\Bigg)+(p\bar{\epsilon}\epsilon\bar{p})(s+t+u)-(p\bar{\epsilon}p)(\bar{p}\epsilon\bar{p})\Bigg)\times \\ \times(\zeta\bar{p})\Bigg(\mathrm{Tr}(\epsilon\bar{\epsilon})\Bigg(-\frac{(s+t)(3s+3t+2u)}{4}\Bigg)-(p\bar{\epsilon}\epsilon\bar{p})(s+t+u)-$$

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$$
\left. - (p\bar{\epsilon}p)(\bar{p}\epsilon\bar{p}) + (p\bar{\epsilon}\epsilon p)(s+t) - 2(p\bar{\epsilon}p)(\bar{p}\epsilon p) \right) +
$$

$$
\left. + (\zeta\epsilon\bar{\epsilon}p)\left(\frac{(s+t)u}{2}\right) - (\zeta\epsilon\bar{p})(p\bar{\epsilon}p)u \right], \tag{7.2}
$$

where we have used the equations of motion and gauge transversality conditions to reduce the result in terms of appropriate invariants for comparison with the string amplitude. For the $R\Omega^2$ terms we use the expansion of the second fundamental form in static gauge (5.7) to get contact interactions of two gravitons with one scalar.

The polarization tensor $\bar{\epsilon}^{\alpha\beta}$ has only world-volume indices and this simplifies the procedure considerably. As an example there is no contribution from the $\hat{R}^2$ term since at least one Ricci scalar will be expanded to linear order of the graviton field $\bar{\epsilon}^{\alpha\beta}$ and therefore it vanishes due to the conditions (6.8). As another example, invariants like $(R_{\alpha\beta ij})^2$ contribute:

$$
\begin{aligned}
R_{\alpha\beta ij}R^{\alpha\beta ij} \to\ & \partial_\alpha X^k R_{k\beta ij}R^{\alpha\beta ij} + \xi_i^\gamma R_{\alpha\beta\gamma j}R^{\alpha\beta ij} + \\
& + \text{all other pull-backs} + X^i\partial_i(R_{\alpha\beta ij}R^{\alpha\beta ij}) \,.
\end{aligned}
$$

The normal frame and Taylor expansions vanish and only the tangent frame pull-back contributes.

Again the contact terms alone do not reproduce the full string amplitude. One has to consider exchange terms as well. The two scalars with one graviton vertex of the Born-Infeld action has derivative corrections $R\Omega^2$, so we need to include scalar exchange diagrams as in (figure 4). Each $R\Omega^2$ term gives a vertex with two scalars and one graviton of sixth order in the momenta. Combined with the mixing vertex from $L_{(Xg)}$ in (4.12) we get exchange terms of the same order as the contact terms. The general form of the exchange terms is:

$$
A_{FT}^{\text{exchange}} = V_{Xg}^i V_{XXg}^j P_{ij} \,, \tag{7.3}
$$

where the vertex V_{XXg}^j comes from expansion to linear order of the Riemann and second fundamental form tensors in $R\Omega^2$ and $R\Omega\,\mathrm{Tr}\,\Omega$.

Combining all of the above contributions for the proposed lagrangian $L^{(p)}$ we get:

$$
\begin{aligned}
A_{FT} \sim\ & \gamma_3\left(\frac{s^2-t^2-u^2}{2}\right) + -2\gamma_2\left(\frac{u^2}{2s+2t+u} + (t-s)\right) + \\
& +\gamma_7\left(\frac{u^2}{4} + (c_2-c_1)\frac{(s+t+u)^2}{2}\right) + \gamma_8(c_1-c_2)(2s+2t) + 2\gamma_9(c_1-c_2) + \\
& +\gamma_1\left(\frac{u^2}{2s+2t+u} + (t-s) + (c_2-c_1)u\right) + \gamma_{10}(c_2-c_1)(2u) \times \\
& \times\gamma_6\left(\frac{4(s+t)(s+t+u)+u^2}{4} + (c_1-c_2)\frac{-2(s+t)^2-4u(s+t)-2u^2}{4}\right) +
\end{aligned}
$$

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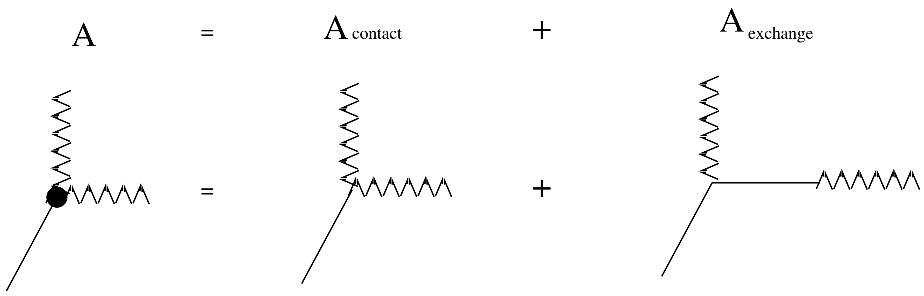

Figure 4: The $\mathcal{O}(\alpha'^2)$ field theory amplitude A from $L^{(p)}$ is the sum of a contact part, $A^{\rm contact}$ and a scalar exchange part, $A^{\rm exchange}$.

$$+2\gamma_{11}(c_2-c_1)+\gamma_{12}(c_1-c_2)(2s+2t)+\gamma_5(2s+2t+u+(c_2-c_1)2u)+$$
$$\gamma_4\left(-\frac{2s+2t+u}{2}+(c_1-c_2)u\right), \tag{7.4}$$

where the form factors are given by (6.12) and the definitions:

$$\begin{aligned}
\gamma_8 &= (\zeta p)(p\bar\epsilon\epsilon\bar p) & \gamma_9 &= (\zeta p)(p\bar\epsilon p)(\bar p\epsilon\bar p)\\
\gamma_{10} &= (\zeta p)(p\bar\epsilon\epsilon p) & \gamma_{11} &= (\zeta\bar p)(p\bar\epsilon p)(\bar p\epsilon\bar p)\\
\gamma_{12} &= (\zeta\bar p)(p\bar\epsilon\epsilon\bar p)\,. & &
\end{aligned} \tag{7.5}$$

We observe that c_5 dropped out of the expression (7.4) since the contact piece canceled against the exchange as in the case of c_4. We can determine c_1 and c_2 by comparison with the string amplitude (6.11). The extra form factors defined in (7.5) are absent in the string result. The conditions for absence of these terms are supplemented by the conditions imposed by the relative coefficients of the form factors in (6.11). As a result, the field theory amplitude reproduces correctly the string amplitude structure if the ambiguous coefficients satisfy the relations:

$$c_1 = c_2\,, \tag{7.6}$$

while as mentioned before:

$$c_5 \to \text{undetermined}\,.$$

Combining the conditions (5.13) and (7.6) we have determined the ambiguous coefficients c_2, c_3 in terms of only one unknown c_1:

$$c_1 = c_2 = c_3\,. \tag{7.7}$$

At this point we should make two observations. As claimed the terms proportional to c_4 and c_5 remain undetermined. This is not obvious a priori although the proposed langrangian (5.2) requires the appearance of such terms. In the next section we will try to give an explanation to the fact that c_4 and c_5 remain undetermined. The second observation has to do with the other three coefficients c_1, c_2 and c_3 which

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are equal to each other. Using (7.7) in (5.4) and in view of the Gauss-Codazzi equations (B.6) and (B.7), they combine to form a "Gauss-Bonnet" type lagrangian for the induced metric:

$$\frac{\sqrt{\tilde{g}}}{32\pi^2}L^{\mathrm{ambig}} = \tilde{L}^{(T)}_{GB} = \frac{\sqrt{\tilde{g}}}{32\pi^2}\Big((R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta} - 4(\hat{R}_T)_{\alpha\beta}(\hat{R}_T)^{\alpha\beta} + (\hat{R}_T)^2\Big)\,, \quad (7.8)$$

where we use $\tilde{L}^{(T)}$ to remind ourselves, that this expression holds up to terms proportional to the trace of the second fundamental form, which should appear in $L^{(T)}_{GB}$ since equations (B.6) and (B.7) dictate such terms for $(\hat{R}_T)_{\alpha\beta}$ and $(\hat{R}_T)$.

The unknown coefficient c_1 has already been determined in [18] using a duality argument. We will shortly review this argument for completeness. Type IIA string theory compactified on a K3 manifold is known to be dual to heterotic string theory on T^4 [27]. Assume now that we have a wrapped D4-brane on the surface K3. It is known that D4-branes form bound states with D0-branes. Actually D4-branes carry D0 charge [26].

Now duality with heterotic string theory leads to the following mass formula on the type-IIA side [28] for a bound state of D4- and D0-branes:

$$m = T_{(4)}n_4 V_{K3} + T_{(0)}n_0\,, \quad (7.9)$$

where n_4 and n_0 are the D4 and D0 charges respectively. The DBI action is proportional to the volume of the surface the brane wraps. This reproduces the first part of the mass formula (7.9). The second part should be reproduced by the R^2 terms of the action. For a brane wrapping on a K3 surface only the first term $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ contributes. The second terms vanishes since K3 is Ricci flat and the last two because the normal bundle is trivial. The Gauss-Bonnet lagrangian is proportional to the $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ term for a Ricci flat manifold. As mentioned before L_{GB} is a topological invariant, the Euler number, for 4d manifolds. For K3 it takes the value:

$$\chi \cong \int_{K3} d^4x L_{GB} = \frac{1}{32\pi^2}\int_{K3} d^4x\sqrt{\tilde{g}}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 24\,. \quad (7.10)$$

Therefore using $L_{(p)}$ from (5.3) in (7.9) we get:

$$m = T_{(4)}V_{K3} + (4\pi^2\alpha')^2 T_{(4)}(c_1+1) = T_{(4)}V_{K3} + T_{(0)}(c_1+1)\,. \quad (7.11)$$

Comparison with (7.9) shows that $c_1 = 0$. This consequently sets the c_2 and c_3 ambiguities to zero as well. We have verified in this way that the lagrangian (5.2) proposed by [18] is indeed correct to order $(\alpha')^2$ up to terms proportional to the trace of the second fundamental form.

8. Conclusions

In this paper, we studied the structure of R^2 terms in the D-brane effective action extending the analysis of [18] for the case of non-geodesic ($\Omega \neq 0$) embed-

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dings. We compared the results of our string amplitude computations with the action proposed in ref.[18] supplemented by an ambiguous langrangian term (5.4). We found that the string amplitudes relate the ambiguous coefficients c_1, c_2, c_3 of (5.4) among each other giving us two equations for the three unknowns. These equations can be used to determine two of the coefficients, say c_2, c_3, in terms of the third one, c_1. Nevertheless, the string amplitudes alone do not fix the complete form of the action, leaving the coefficients (c_4, c_5) and c_1, as mentioned above, undetermined. The conjectured type-IIA–heterotic duality was used to determine the value of c_1. The duality argument holds for branes wrapped on K3, a geometry with trivial normal bundle on the D-brane. Our results though are valid for arbitrary normal bundles. It would be interesting to understand the importance of our corrections for type-IIA–heterotic duality in the case of a nontrivial normal bundle.

Therefore we conclude that the lagrangian proposed in [18] reproduces the $\mathcal{O}(\alpha'^{\,2})$ terms of our string scattering amplitudes. The guess that $R^2, R\Omega^2$ and Ω^4 terms can be written in terms of invariants constructed by the world-volume curvature R_T is indeed correct up to terms which vanish in the lowest order equations of motion for the graviton and scalars. Such terms do not contribute when we consider linear expansion in the fields for the Riemann tensor and second fundamental form. Nevertheless there is no reason a priori that they should not be present in higher point amplitudes where we need to expand the Riemann and second fundamental form tensors beyond the leading order. Nevertheless cancellation of the contact and exchange contributions of those terms makes it impossible to determine their coefficients in the D-brane lagrangian.

As promised earlier we will attempt to explain why the coefficients c_4 and c_5 remain undetermined. The equivalence theorem (see [30, 31]) states that the S-matrix elements are invariant under field redefinitions. In other words although field redefinitions might change some coefficients in the lagrangian, the scattering process does not depend on such terms. This was exactly the case in [29][2] where in the context of heterotic string theory similar cancelations between contact and exchange contributions made it impossible to deteremine through string amplitude computations R^2 corrections that involve $R_{\mu\nu}$ or R. It was actually demonstrated in [29] that even loop computations cannot determine the full structure of the higher derivative terms. The same cancelations between contact and exchange diagrams persist in genus 1 superstring amplitudes and render the fixing of such lagrangian terms impossible. It was actually conjectured that these cancelations will continue at higher loop amplitudes as well.

Following the discussion of [32], if the on-shell action at higher order in some coupling constant g (α'^2 in our case) contains terms proportional to the equations of

[2]We wish to thank Pascal Bain for pointing out [29] to us.

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motion obtained from the lower-order action, as in example:

$$S[\phi] = S_0[\phi] + g \int dx \frac{\delta S_0[\phi]}{\delta\phi(x)} R(\phi(x), \partial\phi(x)) \tag{8.1}$$

these can be removed by the field redefinition:

$$\phi \to \phi - gR(\phi, \partial\phi) \to S[\phi] \to S_0[\phi] + \mathcal{O}(g^2) \tag{8.2}$$

resulting in an ambiguity for such langrangian terms. In our case it is a field redefinition of the scalar field X^i that generates an infinite set of derivative terms proportional to $\mathrm{Tr}\,\Omega$. Under the redefinition:

$$X^i \to X^i + \frac{1}{24}\frac{(4\pi^2\alpha')^2}{32\pi^2}(q\Omega^i_{\alpha\beta}R^{\alpha\beta} + r\Omega^i_{\alpha\beta}\Omega^{j\ \beta\gamma}\Omega^{j\ \alpha}_{\gamma}) \tag{8.3}$$

the effective lagrangian changes:

$$\delta L^{(p)} = \frac{T_{(p)}e^{-\phi}\sqrt{\tilde{g}}}{2}\frac{(4\pi^2\alpha')^2}{24\cdot 32\pi^2}(\Omega^{i\ \alpha}_{\alpha})(q\Omega^i_{\alpha\beta}R^{\alpha\beta} + r\Omega^i_{\alpha\beta}\Omega^{j\ \beta\gamma}\Omega^{j\ \alpha}_{\gamma}) + \mathcal{O}(\alpha'^4)\,, \tag{8.4}$$

where the variation of the lagrangian came from the variation of $\sqrt{\tilde{g}}$ and we have used the formula:

$$\frac{\delta\sqrt{\tilde{g}}}{\delta X^i} = \frac{1}{2}\sqrt{\tilde{g}}\Omega^{i\ \alpha}_{\alpha}\,. \tag{8.5}$$

The above redefinition generates a symmetry of the S-matrix under the shifts: $\delta c_4 = r/2$ and $\delta c_5 = q/2$. As in [29] we expect this phenomenon to continue to higher genus(loop) computations. In other words pertubation theory cannot help us fix the form of the lagrangian completely. We will have to perform some non-perturbative or off-shell computation in order to determine the ambiguous terms.

The whole discussion above does not mean of course that terms subject to change under field redefinitions are unimportant or can be set to any value we choose. As demonstarted in [29] under field redefinitions of the graviton the lagrangians we get correspond to physically inequivalent theories. In their case, depending on the value of the arbitrary coefficients, the R^2 corrections can be of the form $(R_{\mu\nu\rho\sigma})^2$ or a Gauss-Bonnet term or even $(C_{\mu\nu\rho\sigma})^2$, where $C_{\mu\nu\rho\sigma}$ the Weyl tensor. But each choice leads to a different spectrum with the GB case giving a spin-2 graviton, the R^2 case in addition to the spin-2 graviton a scalar and the C^2 case ghost-like fields. In our case although there is no obvious difference in the low energy spectrum for different choices of the coefficients c_4 and c_5, it is in principle plausible that such terms will affect calculations involving branes in curved backgrounds. For example they could proove crucial for studying thermal YM theories employing the AdS/CFT correspondence. The consistency of the computations in [33], which involve D-brane probes approaching the horizon of black holes in AdS space-time, depends on the form of the acceleration terms on the D-brane world-volume action. It is therefore important to determine the complete structure of the higher-derivative terms.

Another explanation[3] for the cancelation of contact and exchange diagrams is

[3]This argument was suggested to us by I. Antoniadis and P. Vanhove.

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that since we expanded the second fundamental form to the the next order in α' then the equations of motion for the scalars receive α' corrections and become $\partial^2 X^i = \mathcal{O}(\alpha')$. Therefore one needs to modify the propagator of the scalar fields. We used the scalar propagator to the zeroth order in α' in our computations. By adding exchange and contact pieces we actualy solved diagrammaticaly the equations of motion for the scalar fields. To this order the equations of motion are still $\mathrm{Tr}\,\Omega^i = 0$ and therefore terms proportional to $\mathrm{Tr}\,\Omega$ do not contribute to our amplitudes.

The results first presented in [18] were used for checking various dualities in cases where the second fundamental form Ω was vanishing. Now, having determined the complete form of the action, it would be interesting to extend this analysis to non-geodesic ($\Omega \neq 0$) manifolds.

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A. Integral formulas 1

The integrals we need to evaluate for the string amplitude (3.1) are of the general form:

$$I = \int_{\mathcal{H}^+} d^2z |1-z|^a |z|^b (z-\bar{z})^c (z+\bar{z})^d \,. \tag{A.1}$$

The region of integration $\mathcal{H}^+$ is the upper half complex plane. This integral is convergent only when

$$\begin{aligned} a+b+c &\leq -2 \\ a+b+d &\leq -2 \,. \end{aligned} \tag{A.2}$$

These conditions are not satisfied for the integrals we need to calculate in section 3. We will calculate the integral in the convergent region and then analytically continue to the physical region.

In order to perform this integration we use the well known trick of converting the integrand to two integrals of gaussian form, using the formulas

$$\begin{aligned} |z|^b &= \frac{1}{\Gamma(-b/2)} \int_0^\infty du\, u^{-b/2-1} e^{-u|z|^2} \\ |1-z|^a &= \frac{1}{\Gamma(-a/2)} \int_0^\infty ds\, s^{-a/2-1} e^{-s|1-z|^2} \,. \end{aligned} \tag{A.3}$$

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By writing the complex integral over $z = x + iy$, as two real integrals over the x and y axis the two integrals decouple. We can use known formulas for gaussian integrals to perform the y integration

$$I_y = \int_0^\infty dy\; y^c e^{-(s+u)y^2} = \frac{\Gamma(\frac{1+c}{2})}{2(s+u)^{\frac{1+c}{2}}}\,. \tag{A.4}$$

The integral over x can be found using the generating functional

$$\begin{aligned} F(\lambda) &= \int_{-\infty}^{\infty} dx e^{-(s+u)x^2+2\lambda x} \\ &= \sqrt{\frac{\pi}{s+u}} e^{\frac{\lambda^2}{s+u}}\,. \end{aligned} \tag{A.5}$$

The integral over x is then given by

$$I_x = 2^d e^{-\frac{us}{s+u}} \int_{-\infty}^{\infty} dx\; x^d e^{-(s+u)(x-\frac{s}{s+u})^2} = e^{-s} \frac{d}{d^d\lambda} F(\lambda)\Big|_{\lambda=s}\,. \tag{A.6}$$

The expression above gives the integral over x as a rational function of s and u with an exponential factor in front. Although we do not have this result in a closed form, all the integrals in our calculations involve $d = n$, $n \in \mathbb{Z}$ which can be easily done for each case separately. The necessary integrals are:

$$I_x = 2^d e^{-\frac{us}{s+u}} \frac{\sqrt{\pi}}{(s+u)^{1/2}} \begin{cases} 1\,, & d = 0 \\ \dfrac{u}{(s+u)}\,, & d = 1\,. \end{cases}$$

Following the method introduced by [34] for calculating the four closed string amplitude on the sphere we can perform the integrals over s, u by making the change of variables

$$\begin{aligned} w &= \frac{u}{s+u}\,, \qquad 0 \le w \le 1 \\ v &= \frac{su}{s+u}\,, \qquad 0 \le w \le \infty \end{aligned} \tag{A.7}$$

The final result is:

$$I = (2\imath)^c 2^d\, \pi \frac{\Gamma(1+d+\frac{b+c}{2})\Gamma(1+\frac{a+c}{2})\Gamma(-1-\frac{a+b+c}{2})\Gamma(\frac{1+c}{2})}{\Gamma(-\frac{a}{2})\Gamma(-\frac{b}{2})\Gamma(2+c+d+\frac{a+b}{2})}\,, \tag{A.8}$$

where $d = 0, 1$

B. Some geometry of submanifolds

We follow the analysis of the geometrical features of submanifolds as described in [18, 37, 38]. To describe the embedding of a D-brane in the ambient space we

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introduce coordinates σ^α, $(\alpha = 0, \ldots, p)$. The fields $X^\mu(\sigma)$, $\mu = 0, \ldots, 9$ describe the embedding of the p-brane in the ten dimensional space-time. The two vector fields $\partial_\alpha X^\mu$ and ξ_i^μ, $(i = p+1, \ldots, 9)$ define tangent and normal bundle frames respectively and satisfy the relations

$$\xi_i^\mu \xi_j^\nu G_{\mu\nu} = \delta_{ij} \qquad \text{and} \qquad \xi_i^\mu \partial_\alpha X^\nu G_{\mu\nu} = 0\,, \tag{B.1}$$

where δ_{ij} is the normal bundle metric. Using these frames we pull-back tensors of the ambient space to the tangent and normal bundle. The pull-back metric has the form

$$\tilde{g}_{\alpha\beta}(\sigma) = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}\,. \tag{B.2}$$

We raise and lower world-volume indices using this metric and similarly, for normal bundle indices using δ_{ij}. There are three independent covariant derivatives defined with respect to: the ambient space connection $\Gamma^\mu_{\nu\rho}$, tangent bundle connection $(\Gamma_T)^\gamma_{\alpha\beta}$ and the normal bundle connection $(\omega)^{ij}_\alpha$

$$(\omega)^{ij}_\alpha = \xi^{\mu,[i}(G_{\mu\nu}\partial_\alpha + G_{\mu\sigma}\Gamma^\mu_{\nu\rho}\partial_\alpha X^\rho)\xi^{\nu,j]}\,, \tag{B.3}$$

which is defined by requiring the normal frame to be covariantly constant. The ambient space and tangent bundle connections are the usual Christoffel symbols of the corresponding metrics.

Differentiating covariantly (B.2) with respect to the world-volume coordinates σ the left hand side vanishes, since it is the covariant derivative of the pull back metric with respect to the world-volume connection. The right hand side is the projection to the tangent bundle of a tensor, which we define as the second fundamental form

$$\Omega^\mu_{\alpha\beta} = \Omega^\mu_{\beta\alpha} = \partial_\alpha\partial_\beta X^\mu - (\Gamma_T)^\gamma_{\alpha\beta}\partial_\gamma X^\mu + \Gamma^\mu_{\nu\rho}\partial_\alpha X^\mu \partial_\beta X^\nu\,. \tag{B.4}$$

This is a vector of the ambient space and a second rank world-volume tensor. However by construction the second fundamental form has a vanishing projection to the tangent bundle therefore we need only to consider the normal bundle projection

$$\Omega^i_{\alpha\beta} = \Omega^\mu_{\alpha\beta}\xi^i_\mu\,. \tag{B.5}$$

There are two ways to construct Riemann tensors which transform covariantly under world-volume reparametrizations and normal frame rotations. One way is to pull-back the ambient space Riemann tensors and the second by constructing Riemann tensors from world-volume and normal bundle connections. However the Gauss-Codazzi equations relate these two kinds of tensors

$$(R_T)_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \delta_{ij}(\Omega^i_{\alpha\gamma}\Omega^j_{\beta\delta} - \Omega^i_{\alpha\delta}\Omega^j_{\beta\gamma}) \tag{B.6}$$

and

$$(R_N)_{\alpha\beta}{}^{ij} = R_{\alpha\beta}{}^{ij} + g^{\gamma\delta}(\Omega^i_{\alpha\gamma}\Omega^j_{\beta\delta} - \Omega^j_{\alpha\gamma}\Omega^i_{\beta\delta})\,. \tag{B.7}$$

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When writing possible $\mathcal{O}(\alpha'^2)$ invariants we need to keep in mind that we cannot determine, by comparison with string amplitudes, terms which vanish due to the lowest-order equations of motion since the string amplitudes are evaluated on-shell. Therefore, since the lowest order equations of motion impose the vanishing of the ambient space Ricci tensor and of the trace of the second fundamental form at linearized level,

$$R_{\mu\nu} = 0 \qquad \text{and} \qquad \Omega^{\mu}{}_{\alpha}{}^{\alpha} = 0 \tag{B.8}$$

invariants involving these tensors are non-vanishing only if we expand them beyond the linear approximation. The $(curvature)^2$ terms determined in [18] are modulo invariants which involve these tensors since in their case all field theory vertices were derived from the linearized form of the curvature and second fundamental form tensors. Expanding around a flat static space-time and using the static gauge for the embedding

$$X^{\mu}(\sigma) = (\sigma^0, \ldots \sigma^p, X^{p+1}(\sigma), \ldots, X^{p+1}(\sigma)) \tag{B.9}$$

equations (B.8) imply the mass-shell conditions on graviton and scalar field respectively. As a result of the first condition in (B.8) we only need to consider Ricci tensors $\hat{R}_{\alpha\beta}$ $\hat{R}_{ij}$ constructed from contractions of world-volume indices of the pull-back curvature tensors. We use the hat on these tensors to distinguish them from pull-backs of Ricci tensors of the ambient space. In addition the possible invariants are constrained by the symmetries of the Riemann tensor and the cyclic permutation property

$$R^{\mu}_{[\nu\rho\sigma]} = 0\,. \tag{B.10}$$

As a result of these constrains not all pull-backs of the ambient space Riemann tensor are independent. Eventually, it turns out that only six pull-backs of the Riemann tensor

$$\begin{aligned} &R_{\alpha\beta\gamma\delta},\ R_{\alpha\beta\gamma i},\ R_{\alpha\beta ij},\\ &R_{\alpha\{ij\}\beta},\ R_{\alpha ijk},\ R_{ijkl}, \end{aligned} \tag{B.11}$$

and four contractions of these pull-backs

$$\hat{R}_{\alpha\beta},\ \hat{R}_{\alpha i},\ \hat{R}_{ij},\ \hat{R}, \tag{B.12}$$

are independent.

The possible $\mathcal{O}(\alpha'^2)$ invariants need to satisfy world-volume reparametrizations and normal frame rotations invariance. Therefore we need to consider full contractions of the tensors enumerated above, among themselves and with the second fundamental form. There are ten R^2, six $R\Omega^2$ and four Ω^4 invariants. The ten R^2 terms are not completely independent with each other, that is, for linear expansion of the $R's$ in the graviton field. There is one ambiguity which is the Gauss-Bonnet term. It turns out that similarly there are ambiguous combinations for the $R\Omega^2$ and

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Ω^4 terms all reducing to total derivatives at linearized level. For completeness we list all the R^2, $R\Omega^2$ and Ω^4 terms: squares of the pull-back Riemann and $\hat{R}$ tensors

$$\begin{array}{llll} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\,, & R_{\alpha\beta\gamma i}R^{\alpha\beta\gamma i}\,, & R_{\alpha\beta ij}R^{\alpha\beta ij}\,, & \\ R_{\alpha\{ij\}\beta}R^{\alpha\{ij\}\beta}\,, & R_{\alpha ijk}R^{\alpha ijk}\,, & R_{ijkl}R^{ijkl}\,, & \\ \hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta}\,, & \hat{R}_{\alpha i}\hat{R}^{\alpha i}\,, & \hat{R}_{ij}\hat{R}^{ij}\,, & \hat{R}^2\,. \end{array} \tag{B.13}$$

and terms involving the second fundamental form

$$\begin{array}{lll} R^{\alpha\beta\gamma\delta}(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})\,, & R^{\alpha\beta}{}_{ij}\,\Omega^i_{\alpha\gamma}\Omega^{j\ \gamma}_{\beta}\,, & R^{\alpha}{}_{\{ij\}}{}^{\beta}\Omega^i_{\alpha\gamma}\Omega^{j\ \gamma}_{\beta}\,, \\ \hat{R}^{\alpha\beta}(\Omega_{\alpha\gamma}\cdot\Omega_{\beta}{}^{\gamma}), & \hat{R}_{ij}\,\Omega^i_{\alpha\beta}\Omega^{j\ \alpha\beta}\,, & \hat{R}\,(\Omega_{\alpha\beta}\cdot\Omega^{\alpha\beta}) \end{array} \tag{B.14}$$

and

$$\begin{array}{ll} (\Omega_{\alpha\beta}\cdot\Omega^{\alpha\beta})(\Omega_{\gamma\delta}\cdot\Omega^{\gamma\delta}), & (\Omega_{\alpha\gamma}\cdot\Omega^{\alpha\delta})(\Omega^{\beta\gamma}\cdot\Omega_{\beta\delta})\,, \\ (\Omega_{\alpha\beta}\cdot\Omega_{\gamma\delta})(\Omega^{\alpha\beta}\cdot\Omega^{\gamma\delta}), & (\Omega_{\alpha\beta}\cdot\Omega_{\gamma\delta})(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta})\,. \end{array} \tag{B.15}$$

The ambiguous combinations which vanish to linear approximation to the fields are

$$\begin{aligned} & R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4\hat{R}_{\alpha\beta}\hat{R}^{\alpha\beta} + \hat{R}^2\,, \\ & 4R_{\alpha\beta\gamma\delta}(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta}) - 8\hat{R}_{\alpha\beta}(\Omega^{\alpha\gamma}\cdot\Omega^{\beta}_{\gamma}) + 2\hat{R}(\Omega^{\alpha\beta}\cdot\Omega_{\alpha\beta}) \times \\ & \times 2(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})(\Omega^{\alpha\gamma}\cdot\Omega^{\beta\delta}) - 2(\Omega_{\alpha\gamma}\cdot\Omega_{\beta\delta})(\Omega^{\alpha\delta}\cdot\Omega^{\beta\gamma})\,, \\ & -4(\Omega_{\alpha\gamma}\cdot\Omega^{\gamma}_{\beta})(\Omega^{\alpha\gamma}\cdot\Omega^{\beta}_{\gamma}) + (\Omega^{\alpha\beta}\cdot\Omega_{\alpha\beta})(\Omega^{\gamma\delta}\cdot\Omega_{\gamma\delta})\,. \end{aligned} \tag{B.16}$$

In [18] the various invariants were combined using the Gauss-Codazzi equations, to quadratic invariants of (R_T) and (R_N). We rewrite the expression involving $(R_T)_{\alpha\beta}$, for the lagrangian of [18], in terms of the invariants listed above. We do so, because it is more convenient to identify the vertices, which contribute to the field theory amplitudes, with the lagrangian in this form. The amplitudes of section 5 and 7 require that the second fundamental form is expanded to the subleading order. In the static gauge this is equivalent to a graviton field as it is shown in section 5. At this point we need to consider two more invariants, which involve the trace of the second fundamental form.

$$(\Omega^{\gamma}_{\gamma}\cdot\Omega^{\beta}_{\alpha})(\Omega^{\alpha}_{\delta}\cdot\Omega^{\delta}_{\beta})\,, \qquad \hat{R}_{\alpha\beta}(\Omega^{\alpha\beta}\cdot\Omega^{\gamma}_{\gamma})\,. \tag{B.17}$$

Finally, expanding pull-backs of Riemann tensors we need the expressions of the tangent and normal bundle frames in terms of open string modes (scalar fields). Although the tangent bundle frame is written explicitly in terms of the fields in the Born-Infeld action

$$\partial_\alpha X^\mu = \delta^\mu_\alpha + \delta^\mu_i\partial_\alpha X_i \tag{B.18}$$

the normal bundle frame is not. We can use equations (B.1) to solve perturbatively for ξ^μ_i. We assume that ξ^μ_i has convergent expansion in powers of the fields X^i which

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represent small fluctuations of the position of the brane in the transverse space. We rescale the fields $X^i \to \lambda X^i$, where λ a small parameter. Consequently, we write ξ_i^μ as an expansion in powers of λ

$$\xi_i^\mu = \xi_{i|0}^\mu + \lambda \xi_{i|1}^\mu + O(\lambda^2)\,. \tag{B.19}$$

By plugging this expansion in to the second of (B.1) and retaining terms up to λ we have

$$\xi_{\alpha|0}^i + \lambda(\partial_\alpha X^j \xi_{j|0}^i + \xi_{\alpha|1}^i) + O(\lambda^2) = 0 \tag{B.20}$$

and the first of (B.1) implies

$$\xi_{i|0}^k \xi_{j|0}^l \delta_{kl} + O(\lambda^2) = \delta_{ij}\,. \tag{B.21}$$

Solving these two equations we get the following solution to subleading order in λ

$$\xi_i^\mu = \delta_i^\mu - \lambda \delta_\alpha^\mu \partial^\alpha X_i + O(\lambda^2)\,. \tag{B.22}$$

In our analysis, of the permissible invariants that contribute to the graviton and three scalars amplitude, we have excluded terms of the type D^2R, $\Omega D^2\Omega$, $\Omega^2 D\Omega$ and ΩDR which are corrections to the one-point function of the graviton, scalar propagator and graviton-scalar mixing. These vertices are presumably protected by supersymmetry [18] and do not receive derivative corrections. It can also be checked explicitly that such terms make unacceptable contributions to the string amplitudes considered in [18] as well as to our amplitudes.

C. Integral formulas 2

The integrals we have to compute in section 6 are of the general form:

$$I(a,b,c;P(z,\bar{z})) = \int_{|z|<1} d^2z |1-z|^{2a} |z|^{2b} (1-|z|^2)^c \; P(z,\bar{z})\,, \tag{C.1}$$

where $P(z,\bar{z})$ is a polynomial in $z, \bar{z}$. Using polar coordinates the expression above can be written:

$$\int_0^1 rdr \int_0^{2\pi} d\theta (1-\cos\theta + r^2)^a r^{2b} (1-r^2)^c \; P(r\cos\theta, r\sin\theta)\,. \tag{C.2}$$

We can compute the integrals over θ term by term for each polynomial P using formulas from [35]. Our amplitude involves only polynomials up to first power in $r\cos\theta$ so we can use for the integral over θ:

$$\int_0^{2\pi} d\theta \cos n\theta \; (1-2r\cos\theta + r^2)^a = 2\pi \frac{\Gamma(n-a)}{\Gamma(-a)} \frac{r^n}{n!} F_{21}(-a, n-a; n+1; r^2)\,, \tag{C.3}$$

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where F_{21} the hypergeometric function and $n = 0, 1$. Using tabulated formulas from [36] the integral over r takes the general form:

$$\int_0^1 drr^{2b+1+n}(1-r^2)^c\, F_{21}(-a, n-a; n+1; r^2) = \tag{C.4}$$
$$= \frac{1}{2}\frac{\Gamma(1+\frac{2b+n}{2})\Gamma(1+c)}{\Gamma(2+c+\frac{2b+n}{2})}F_{32}\left(-a, n-a, 1+\frac{2b+n}{2}; n+1, 2+c+\frac{2b+n}{2}; 1\right),$$

where the generalized hypergeometric function F_{32} is defined in [35]. Combining the above results we get the integral formula:

$$I(a, b, c; (r\cos\theta)^n) = \tag{C.5}$$
$$= \frac{\pi}{n!}\frac{\Gamma(n-a)}{\Gamma(-a)}\frac{\Gamma(1+\frac{2b+n}{2})\Gamma(1+c)}{\Gamma(2+c+\frac{2b+n}{2})}F_{32}\left(-a, n-a, 1+\frac{2b+n}{2}; n+1, 2+c+\frac{2b+n}{2}; 1\right).$$

Define the following integrals:

$$\begin{aligned}
I_1 &= I(-s-t-u-1, t-1, u; z+\bar{z}) = 2I(-s-t-u-1, t-1, u; r\cos\theta)\\
I_2 &= I(-s-t-u-1, t-1, u+1; 1)\\
I_3 &= I(-s-t-u, t-1, u; 1)\\
&\quad I(-s-t-u-1, t-1, u; (1+z)(1+\bar{z})) = 2I_1 + I_3\,.
\end{aligned} \tag{C.6}$$

The integrals of (6.10) can be evaluated using the expressions above.

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